COMPLEXITY, SCALING, AND A PHASE TRANSITION

4/29/2026

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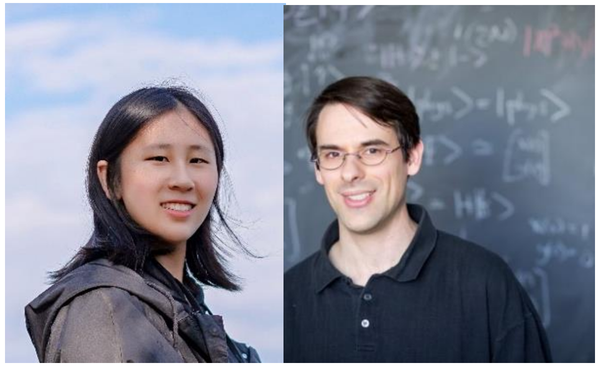

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•2307.08229 [hep-th]

Outline

- Motivation
- Holographic complexity proposals
- Introduction to magnetized AdS soliton
- Evaluation for holographic complexity
- Conclusion

MOTIVATION

Motivation

Quantum mechanics

General Relativity

incompatible

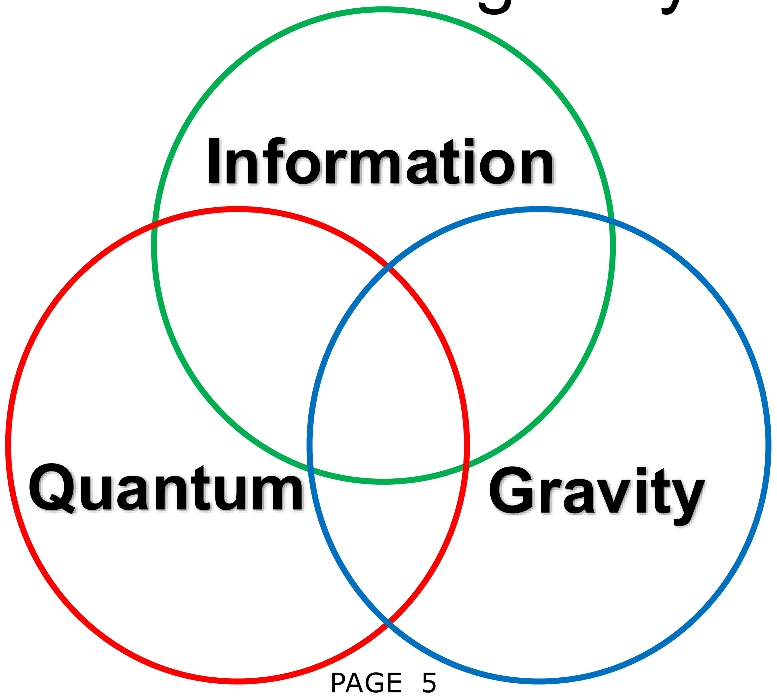
Physicists have undertaken extensive efforts to reconcile them.

Quantum Gravity

- String theory
- Loop quantum gravity
- Black hole thermodynamics
- Holography
- ……

Motivation

- Status: despite extensive efforts, a complete theory of quantum gravity has not yet been found!
- Goal: this presentation aims to explore quantum gravity and gain insights from quantum information in gravity.



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HOLOGRAPHIC COMPLEXITY PROPOSALS

CV, CV2.0, CA

Holography

- Bulk

Anti-de Sitter space theory in d+1 dimension

- Boundary

Conformal field theory in d dimension

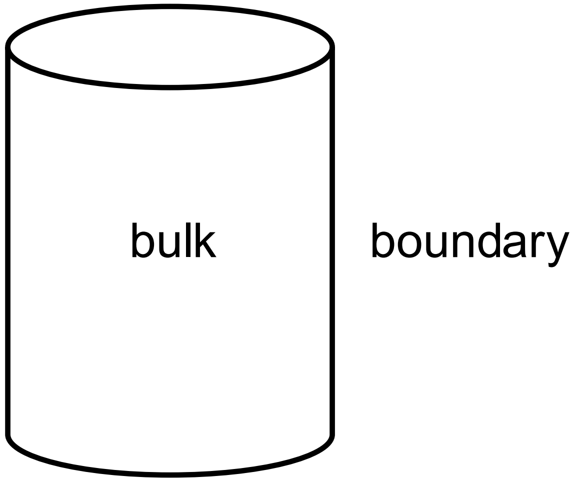
- Correspondence

Holography:

AdS is one dimension higher than CFT.

Duality:

Quantities that can be computed in one theory can also be computed in the dual.



arXiv:gr-qc/9310026, arXiv:hep-th/9409089, arXiv:hep-th/9711200, arXiv:hep-th/9802150 (Hooft, Susskind, Maldacena, and Witten)

Entanglement is Not Enough

- What makes quantum physics so different from classical physics?

1. Entanglement

Knowing everything about a system might reveal nothing about its individual parts.

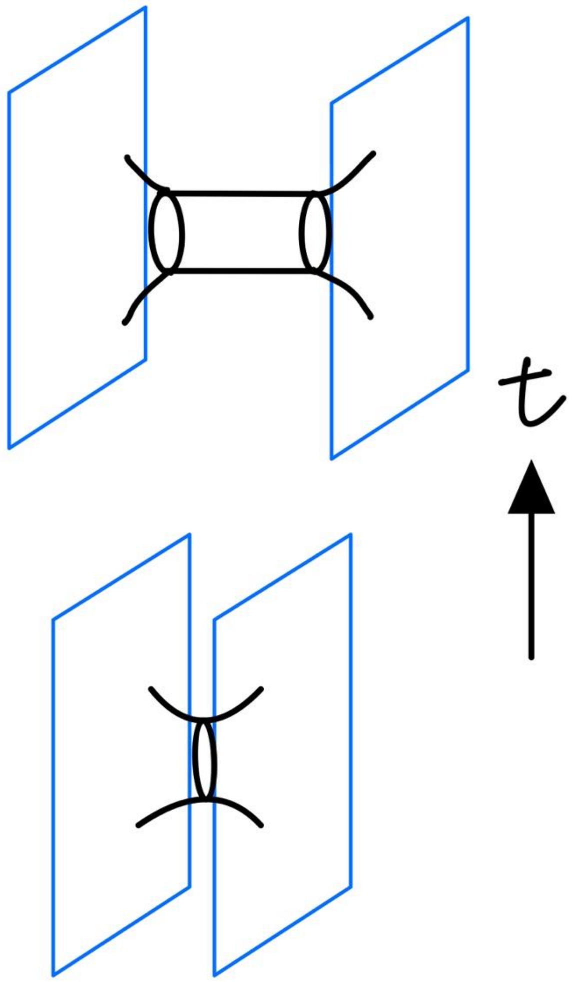
2. Complexity

As Feynman noted, the remarkable complexity inherent in quantum states.

arXiv:1411.0690 (Susskind)

Entanglement is Not Enough

- On the one hand, the ER grows linearly with time for a very long time.
- On the other hand, the state comes to thermal equilibrium quickly, entropy is saturated.
- So entanglement entropy is not enough to understand the rich geometric structures behind the horizon.



Entanglement is not Enough

Leonard Susskind

This is the written version of a lecture given at KITP in Oct 2014 on Black Holes and quantum complexity. I've included (in boldface) various questions that came up during the lecture and discussions the following day, as well as the quantitative calculations that form the basis of the arguments.

Comments: 54 pages, 22 figures
Subjects: **High Energy Physics - Theory (hep-th)**; Quantum Physics (quant-ph)
Cite as: arXiv:1411.0690 **[hep-th]**
(or arXiv:1411.0690v1 **[hep-th]** for this version)
https://doi.org/10.48550/arXiv.1411.0690

arXiv:1411.0690 (Susskind)

Entanglement is Not Enough

- We need something else that keeps growing long after equilibrium.
- The complexity of the state also grows linearly with time at late times.
- Complexity in quantum information/computing theory can capture the growth of the ER!

Computational Complexity and Black Hole Horizons

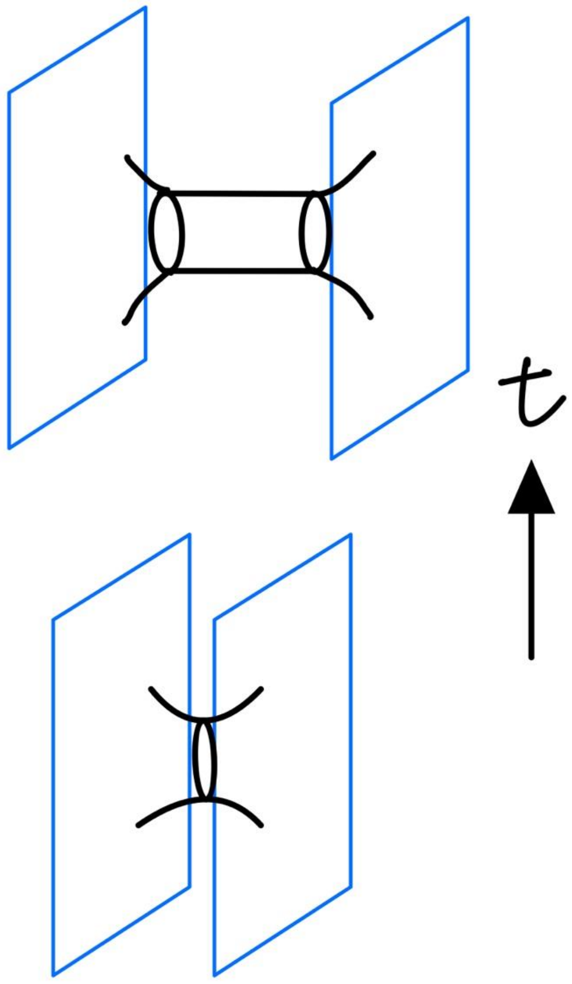
Leonard Susskind

Computational complexity is essential to understanding the properties of black hole horizons. The problem of Alice creating a firewall behind the horizon of Bob's black hole is a problem of computational complexity. In general we find that while creating firewalls is possible, it is extremely difficult and probably impossible for black holes that form in sudden collapse, and then evaporate. On the other hand if the radiation is bottled up then after an exponentially long period of time firewalls may be common. It is possible that gravity will provide tools to study problems of complexity; especially the range of complexity between scrambling and exponential complexity.

Complexity and Shock Wave Geometries

Douglas Stanford, Leonard Susskind

In this paper we refine a conjecture relating the time-dependent size of an Einstein-Rosen bridge to the computational complexity of the of the dual quantum state. Our refinement states that the complexity is proportional to the spatial volume of the ERB. More precisely, up to an ambiguous numerical coefficient, we propose that the complexity is the regularized volume of the largest codimension one surface crossing the bridge, divided by $G_N l_{AdS}$. We test this conjecture against a wide variety of spherically symmetric shock wave geometries in different dimensions. We find detailed agreement.



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arXiv:1402.5674, arXiv:1403.5695, arXiv:1406.2678 (Susskind and Stanford 2014)

Introduction to complexity

- What is complexity?

➢ How hard is it to reach a goal?

➢ How to measure the difficulty to solve a problem？

➢ How many steps you need to take to prepare a given state from a reference state?

In quantum circuit model, quantum complexity is defined as the minimal number of quantum gates (elementary operations) required to reach a target state from the reference sate.

$$|\psi\rangle = U\,|\psi_0\rangle = g_n g_{n-1} \cdots g_1\,|\psi_0\rangle$$

arXiv:1810.11563 (Susskind)

Complexity= Volume conjecture

- Hint

a) The volume of black hole interior grows linearly with time

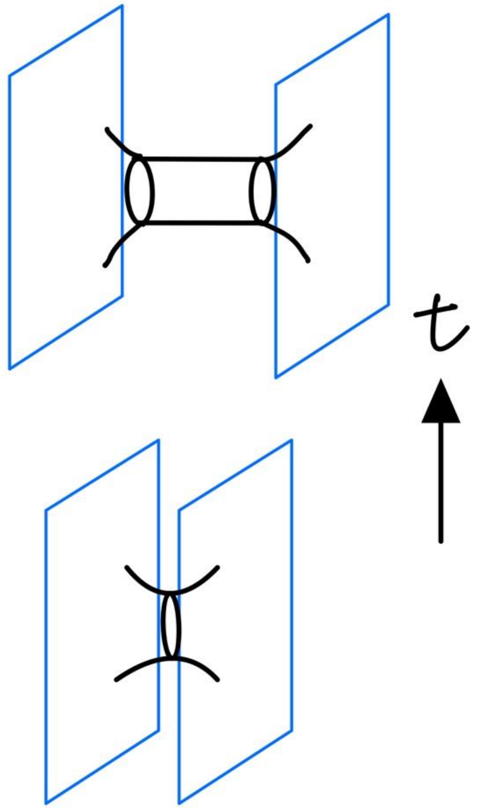
$$\frac{dV}{dt} \propto lAT$$

b) The complexity of the state also grows linearly with time

$$\frac{dC}{dt} = ST$$

c) Thus the complexity is proportional to volume

$$C = STt \propto \frac{A}{G_N} Tt \propto \frac{V}{G_N l}$$



where A is the horizon area, T is the temperature, l is the AdS length scale.

arXiv:1810.11563 (Susskind)

Complexity= Volume conjecture

arXiv > hep-th > arXiv:1810.11563

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High Energy Physics - Theory

[Submitted on 27 Oct 2018]

Three Lectures on Complexity and Black Holes

Leonard Susskind

Given at PiTP 2018 summer program entitled "From Qubits to Spacetime." The first lecture describes the meaning of quantum complexity, the analogy between entropy and complexity, and the second law of complexity.
Lecture two reviews the connection between the second law of complexity and the interior of black holes. I discuss how firewalls are related to periods of non-increasing complexity which typically only occur after an exponentially long time.
The final lecture is about the thermodynamics of complexity, and "uncomplexity" as a resource for doing computational work. I explain the remarkable power of "one clean qubit," in both computational terms and in space-time terms.
The lectures can also be found online at \url{this https URL} .

Comments: 83 pages, 42 figures. This is the written version of three lectures on complexity and black holes
Subjects: **High Energy Physics - Theory (hep-th)**
Cite as: arXiv:1810.11563 **[hep-th]**
(or arXiv:1810.11563v1 **[hep-th]** for this version)
https://doi.org/10.48550/arXiv.1810.11563

Submission history

From: Leonard Susskind [view email]
[v1] Sat, 27 Oct 2018 00:18:27 UTC (3,308 KB)

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arXiv:1810.11563 (Susskind)

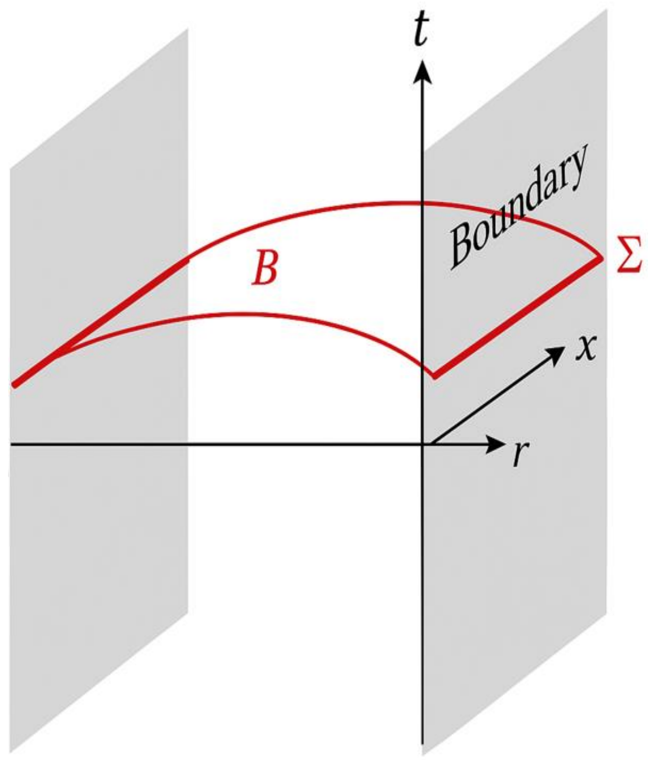
Complexity=Volume conjecture (CV)

- CV Conjecture

The complexity of the boundary CFT state is proportional to the volume of the maximal volume slice.

$$C_V = \frac{(d-1)V}{2\pi^2 G l}$$

- G is the Newton's gravitational constant
- d is the dimension of the boundary CFT
- l is the length scale



Sketch of CV

arXiv:1402.5674, arXiv:1403.5695, arXiv:1406.2678 (Susskind and Stanford 2014)

Complexity= Volume conjecture

- Steps to use CV conjecture

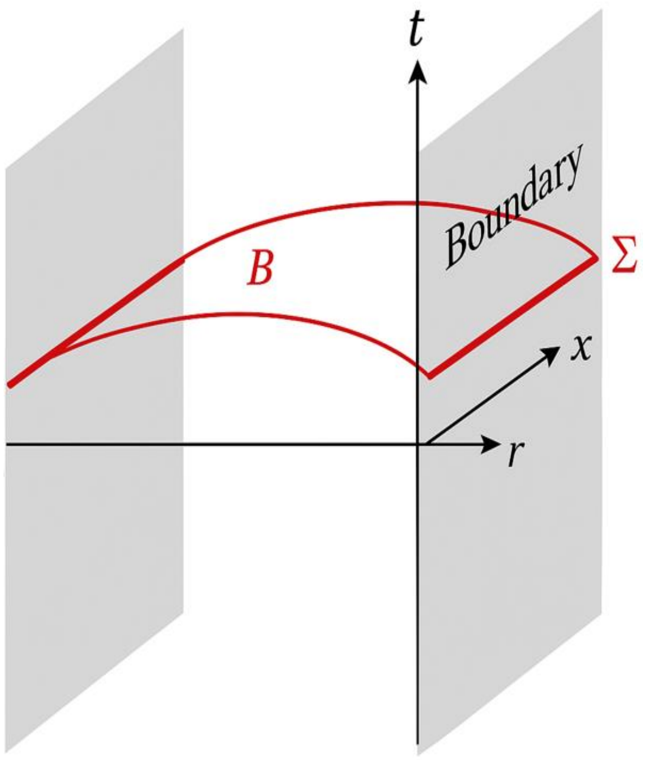
a) Choose a time slice Σ on boundary

b) Consider all spatial B with ∂B = Σ

c) Find the maximum volume

d) Evaluate the complexity

$$C = \frac{(d-1)V}{2\pi^2 G_N l} = \frac{8(d-1)V}{\pi l}$$



Sketch of CV

arXiv:1402.5674, arXiv:1403.5695, arXiv:1406.2678 (Susskind and Stanford 2014)

Complexity=Volume conjecture 2.0 (CV 2.0)

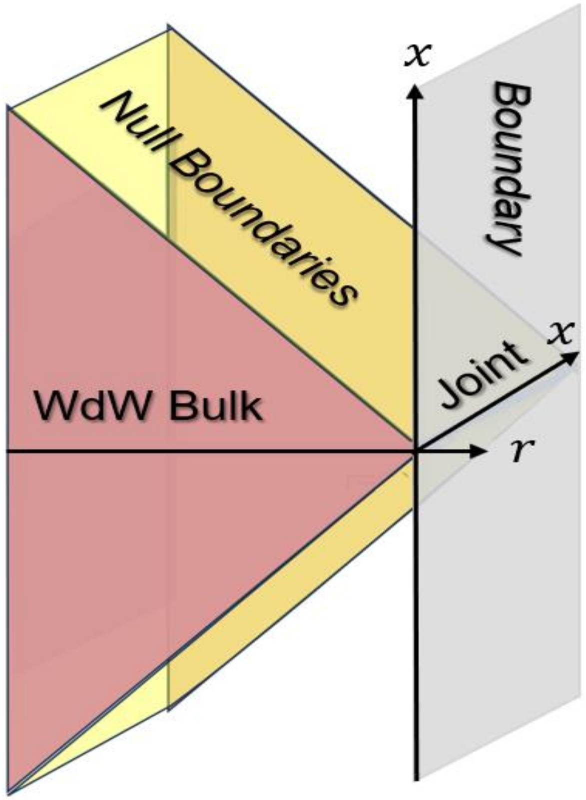
- CV 2.0 conjecture / Complexity=Spacetime volume

The complexity of the boundary CFT state is proportional to the spacetime volume of the Wheeler-de Witt patch.

$$C_2 = \frac{V_{WDW}}{Gl^2}$$

Wheeler-de Witt (WDW) patch:

The region formed by the union of all the possible spacelike hypersurfaces anchored at a fixed boundary.



Sketch of CA

arXiv:1610.02038 (Couch, Fischler, Nguyen 2016)

Complexity=Action conjecture (CA)

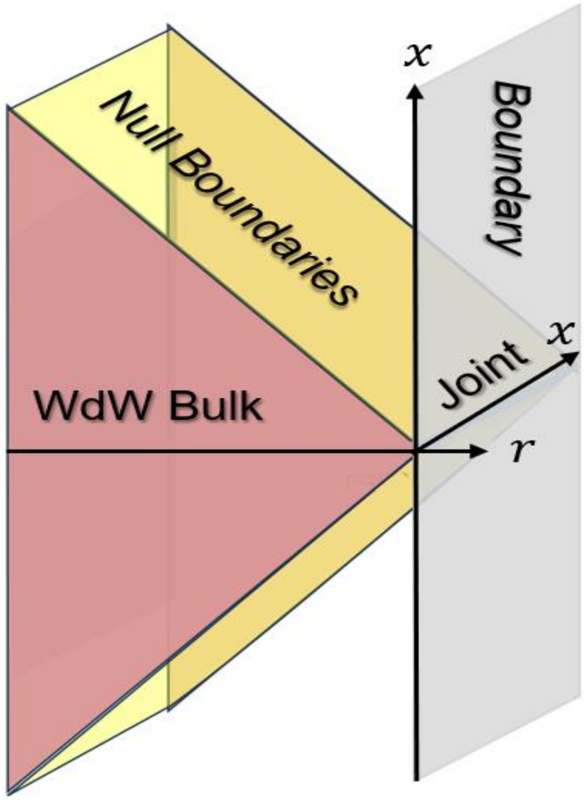
- CA Conjecture

The complexity of the boundary CFT state is proportional to the action of the Wheeler-de Witt patch

$$C_A = \frac{S}{\pi \hbar}$$

S is the action of the Wheeler-de Witt (WDW) patch, the union of all spacelike slices we considered before.

$$S = S_{bulk} + S_{bdy} + S_{joint}$$



Sketch of CA

TY
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arXiv:1509.07876, arXiv:1512.04993 (Brown, Roberts, Susskind, Swingle, Zhao 2015)

Complexity= Spacetime volume or Action conjecture

- Steps to use CV 2.0 or CA conjecture

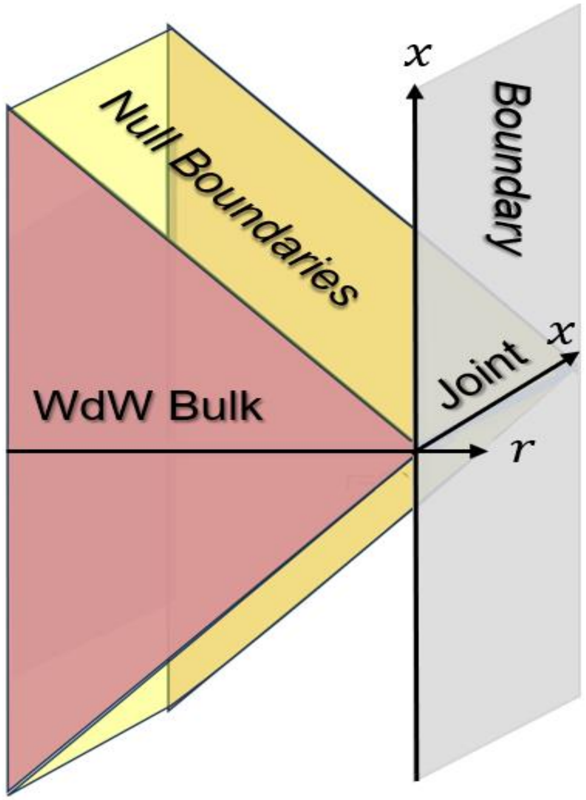
a) Choose a time slice Σ on boundary

b) Consider all spatial B with ∂B = Σ

c) Find the the union of all B, WdW patch

d) Compute the spacetime volume or action of WdW patch

e) Evaluate the complexity



Sketch of CA

arXiv:1509.07876, arXiv:1512.04993 (Brown, Roberts, Susskind, Swingle, Zhao 2015)

INTRODUCTION TO MAGNETIZED ADS SOLITON

Magnetized AdS soliton v.s. periodic AdS

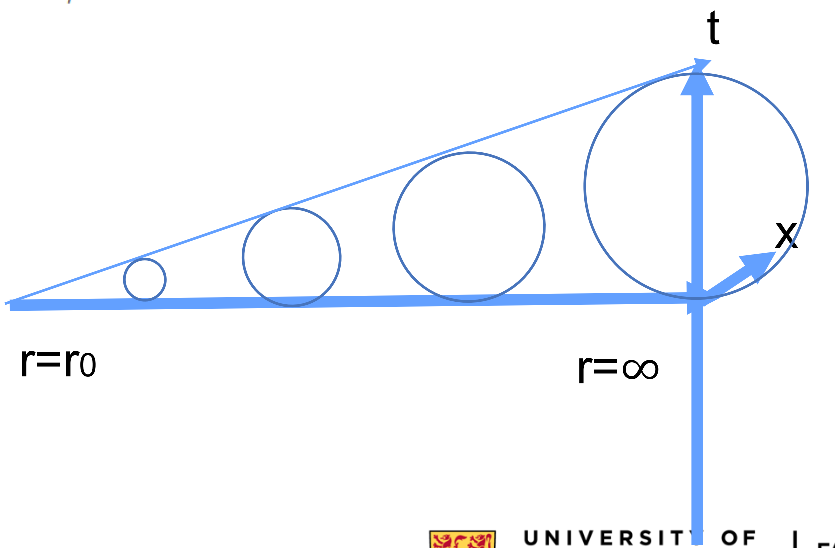
Magnetized AdS soliton

We consider CFTs (for d = 3, 4) compactified on a circle with a U(1) gauge field, which is dual to magnetized AdS$_{d+1}$ soliton bulk

$$ds^2 = \frac{r^2}{l^2}\left(-dt^2 + d\vec{x}^2 + f(r)d\phi^2\right) + \frac{l^2}{r^2 f(r)}dr^2, \qquad f(r) \equiv 1 - \frac{\mu l^2}{r^d} - \frac{Q^2 l^2}{r^{2d-2}}$$

$$A = \sqrt{7-d}\,Q\left(\frac{1}{r^{d-2}} - \frac{1}{r_0^{d-2}}\right)d\phi, \qquad Q \equiv \frac{1}{\sqrt{7-d}}\frac{r_0^{d-2}\Phi}{\Delta\phi}.$$

- A is the gauge potential
- μ is related to Mass/Energy
- Q is the U(1) charge
- Φ is the magnetic flux



arXiv:1712.03732, arXiv:hep-th/9808079, arXiv:hep-th/9803131(Witten,Horowitz, Myers, Reynolds, and Ross)

Magnetized AdS soliton

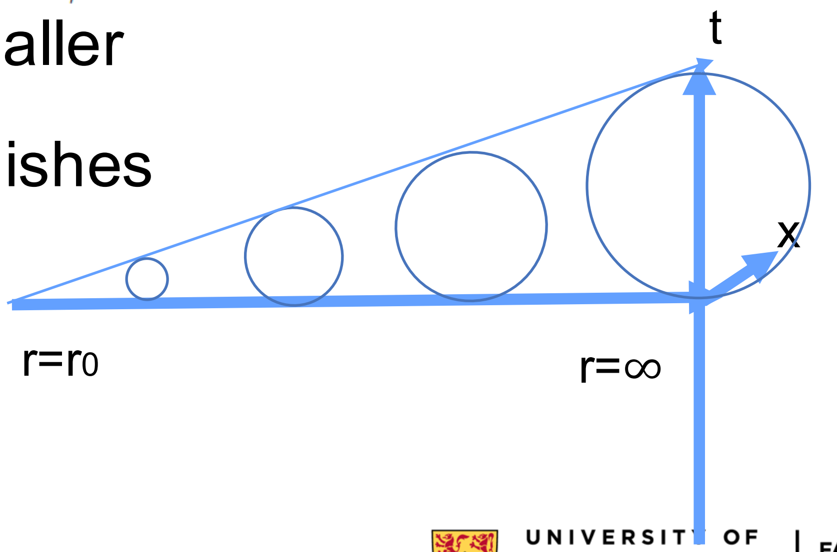
We consider CFTs (for d = 3, 4) compactified on a circle with a U(1) gauge field,

which is dual to magnetized AdS$_{d+1}$ soliton bulk

$$ds^2 = \frac{r^2}{l^2}\left(-dt^2 + d\vec{x}^2 + f(r)d\phi^2\right) + \frac{l^2}{r^2 f(r)}dr^2, \qquad f(r) \equiv 1 - \frac{\mu l^2}{r^d} - \frac{Q^2 l^2}{r^{2d-2}}$$

$$A = \sqrt{7-d}\,Q\left(\frac{1}{r^{d-2}} - \frac{1}{r_0^{d-2}}\right)d\phi, \qquad Q \equiv \frac{1}{\sqrt{7-d}}\frac{r_0^{d-2}\Phi}{\Delta\phi}.$$

- As r decreases, the proper size of ϕ circle get smaller
- At $r = r_0$ (the largest zero of $f(r)$), the circle vanishes
- ϕ has a period $\Delta\phi = \dfrac{4\pi l^2}{r_0^2 f'(r_0)}$



where

$$\mu = \frac{r_0^{2d-2} - Q^2 l^2}{l^2 r_0^{d-2}}, \quad r_0 = \frac{2\pi l^2}{d\Delta\phi}\left(1 + \sqrt{1 - \frac{\Phi^2}{\Phi_{max}^2}}\right)$$

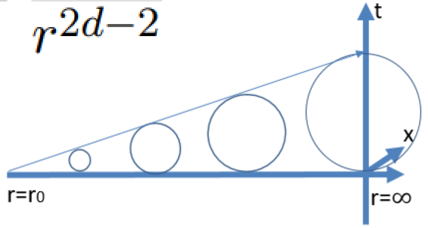
arXiv:1712.03732, arXiv:hep-th/9808079, arXiv:hep-th/9803131(Witten,Horowitz, Myers, Reynolds, and Ross)

Magnetized AdS soliton

AdS soliton

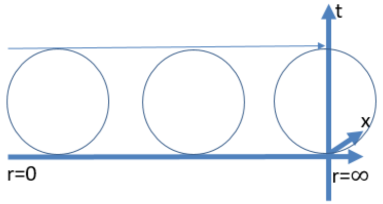
$$ds^2 = \frac{r^2}{l^2}\left(-dt^2 + d\vec{x}^2 + f(r)d\phi^2\right) + \frac{l^2}{r^2 f(r)}dr^2, \qquad f(r) \equiv 1 - \frac{\mu l^2}{r^d} - \frac{Q^2 l^2}{r^{2d-2}}$$

$$A = \sqrt{7-d}\, Q\left(\frac{1}{r^{d-2}} - \frac{1}{r_0^{d-2}}\right)d\phi, \qquad Q \equiv \frac{1}{\sqrt{7-d}}\frac{r_0^{d-2}\Phi}{\Delta\phi}.$$

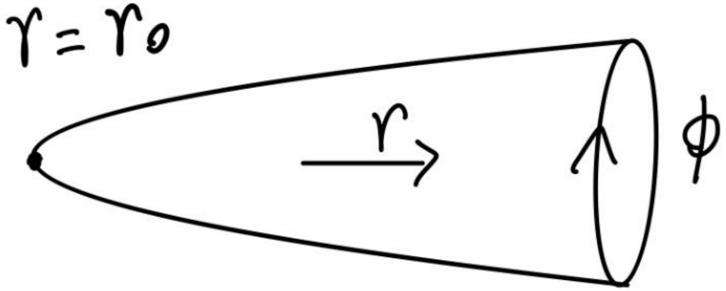


Periodic AdS

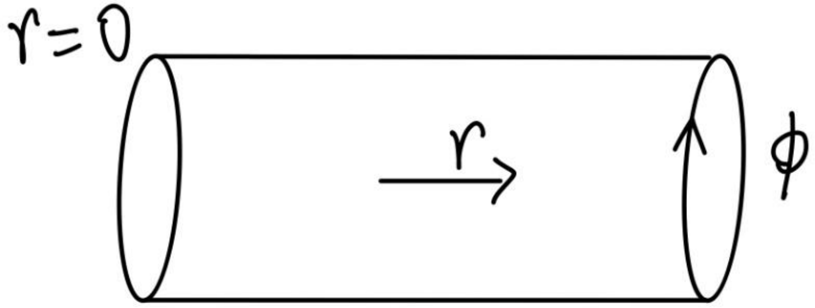
$$ds^2 = \frac{r^2}{l^2}\left(-dt^2 + d\vec{x}^2 + d\phi^2\right) + \frac{l^2}{r^2}dr^2, \qquad A = -\frac{\Phi}{\Delta\phi}d\phi$$



AdS soliton: Cigar-like geometry



AdS: Cylinder-like geometry

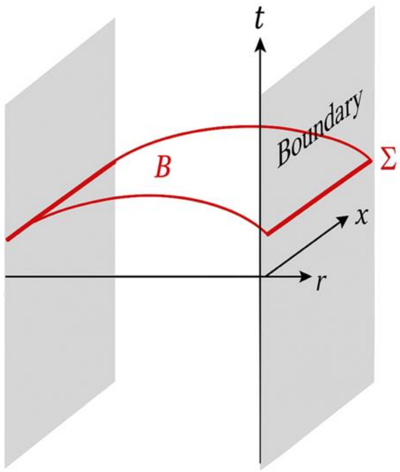


arXiv:1712.03732, arXiv:hep-th/9808079, arXiv:hep-th/9803131(Witten,Horowitz, Myers, Reynolds, and Ross)

EVALUATION FOR HOLOGRAPHIC COMPLEXITY

CV, CV2.0, CA

Evaluation for CV: Complexity=Volume



- The volume of the maximal volume slice is

$$V = V_{\vec{x}} \Delta\phi \int_{r_0}^{r_m} dr \left(\frac{r}{l}\right)^{d-2} = \frac{V_{\vec{x}} \Delta\phi}{d-1} \frac{r_m^{d-1} - r_0^{d-1}}{l^{d-2}}$$

with $r_0 \rightarrow 0$ for periodic AdS, where Vx is the volume along the x directions.

- The CV complexity is proportional to volume

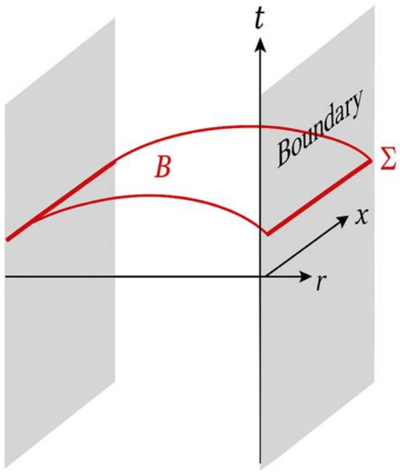
$$C = \frac{(d-1)V}{2\pi^2 G_N l}$$

- Complexity of formation:

Complexity of formation is defined as complexity of the magnetized AdS solitons subtract the corresponding complexity of periodic AdS.

Note： the divergent terms in the maximal volumes cancel as $r_m \rightarrow \infty$

Evaluation for CV : Complexity=Volume



- The volume of the maximal volume slice is

$$V = V_{\vec{x}} \Delta\phi \int_{r_0}^{r_m} dr \left(\frac{r}{l}\right)^{d-2} = \frac{V_{\vec{x}} \Delta\phi}{d-1} \frac{r_m^{d-1} - r_0^{d-1}}{l^{d-2}}$$

with $r_0 \rightarrow 0$ for periodic AdS, where Vx is the volume along the x directions.

- The CV complexity is proportional to volume

$$C = \frac{(d-1)V}{2\pi^2 G_N l}$$

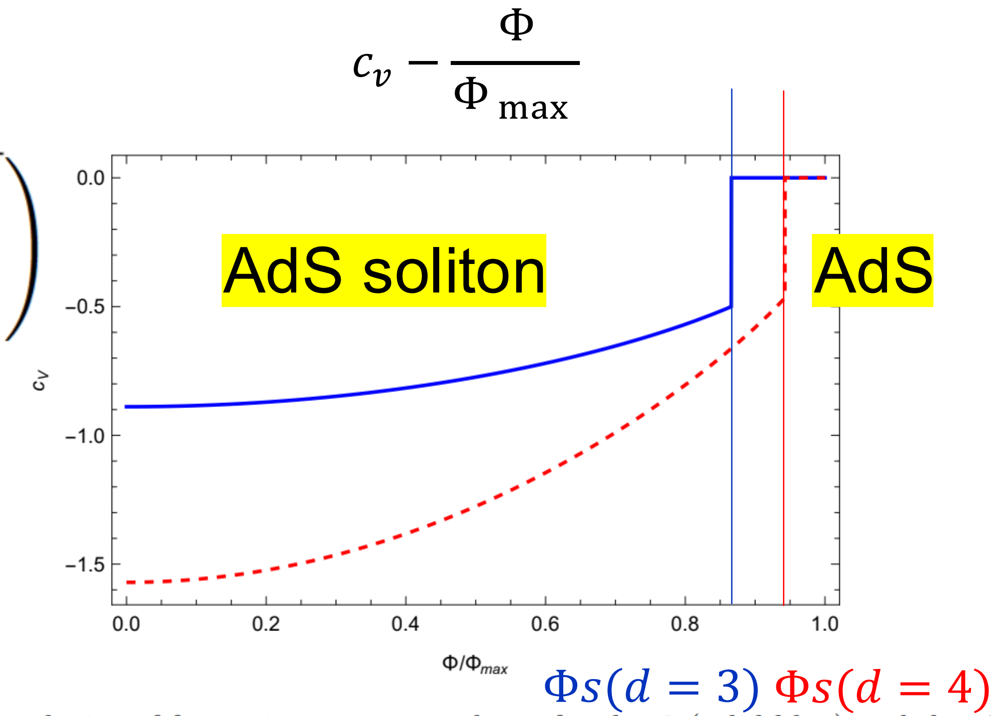
- Complexity of formation per boundary volume is

$$\mathcal{C}_V = -\frac{r_0^{d-1}}{2\pi^2 G l^{d-1}} \qquad r_0 = \frac{2\pi l^2}{d\Delta\phi}\left(1 + \sqrt{1 - \frac{\Phi^2}{\Phi_{max}^2}}\right)$$

Evaluation for CV : Complexity=Volume

CV complexity of formation density is

$$\mathcal{C}_V = -\frac{r_0^{d-1}}{2\pi^2 G l^{d-1}} \qquad r_0 = \frac{2\pi l^2}{d\Delta\phi}\left(1+\sqrt{1-\frac{\Phi^2}{\Phi_{max}^2}}\right)$$



- Scaling

$$\mathcal{C}_V \propto \Delta\phi^{-(d-1)}$$

- Phase transition

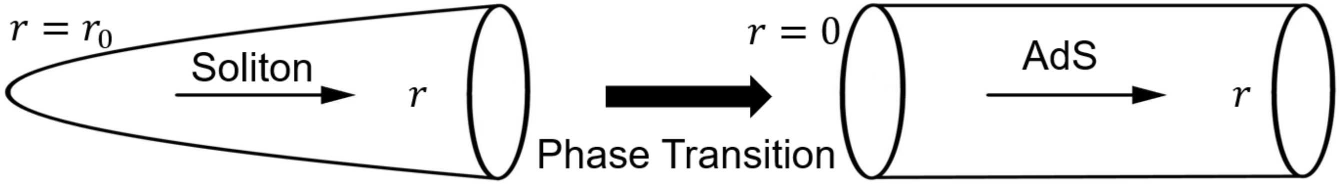
➢ AdS soliton / periodic AdS

➢ Order parameter

non-zero in AdS soliton

zero in periodic AdS!

Figure 1. CV complexity of formation per unit volume for $d = 3$ (solid blue) and $d = 4$ (dashed red). Plotted curves are $c_V \equiv G\mathcal{C}_V(\Delta\phi/l)^{d-1}$, so complexity scales with an inverse power of $\Delta\phi$.



Evaluation for CV 2.0: Complexity=Spacetime volume

- The future and past lightsheets on the WDW patch boundary are given by $t_F(r)$ and $t_P(r)$
- We choose the boundary conditions that both lightsheets satisfy $t = 0$ at the radial cutoff $r = r_m$

Shape of future lightsheet $t_F(r)$

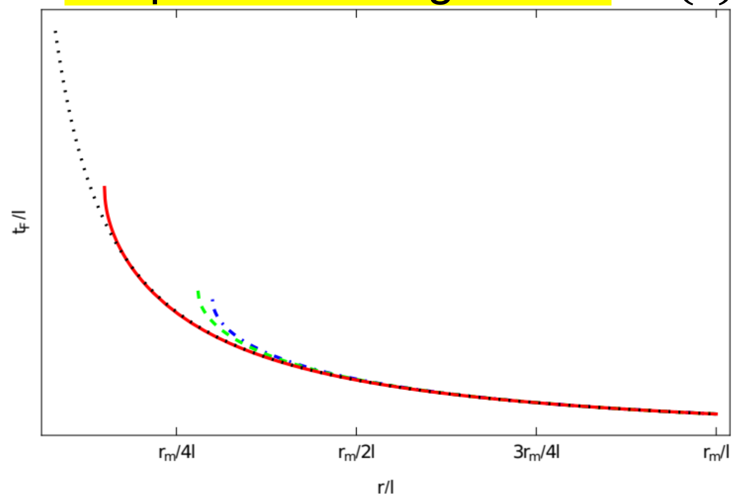
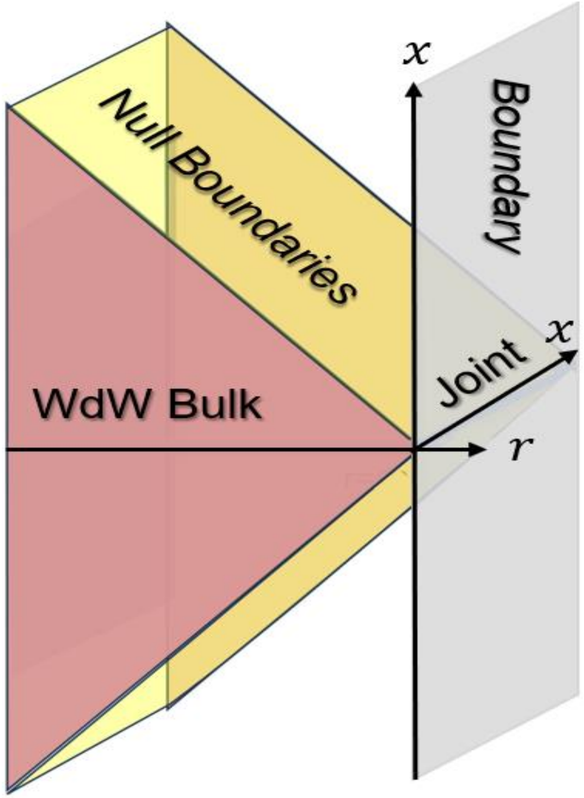


Figure 2. The future lightsheets $t_F(r)$ for $\Phi/\Phi_{\max} = 0$ (dot-dashed blue), 0.5 (dashed green), 1 (solid red). In this instance, $\Delta\phi\, r_m/\ell^2 = 10\pi/3$ and $d = 4$. For reference, $t_F(r)$ for periodic AdS is dotted black.



Sketch of WDW

Evaluation for CV 2.0 : Complexity=Spacetime volume

- The WDW patch volume is

$$V_{\text{WDW}} = \int_{\text{WDW}} d^{d+1}x\sqrt{-g} = 2V_{\vec{x}}\Delta\phi \int_{r_0}^{r_m} dr \left(\frac{r}{l}\right)^{d-1} t_F(r)$$

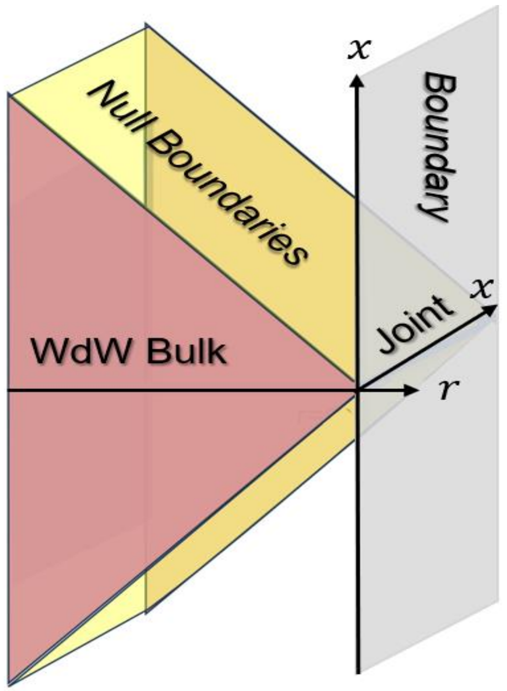
where

$$t_F(r) = \frac{l^2}{r_0}\int_{\tilde{r}}^{r_m/r_0} \frac{d\tilde{r}'}{\tilde{r}'^2\sqrt{f(\tilde{r}')}} \equiv \frac{l}{r_0}\tilde{r}_F(\tilde{r}) \qquad \tilde{r} = r/r_0$$

- The complexity 2.0 is $\quad C_2 = \dfrac{V_{WDW}}{Gl^2}$

- Complexity of formation density

$$\mathcal{C}_2 = \frac{2}{G}\left(\frac{r_0}{l}\right)^{d-1}\left\{-\frac{1}{d(d-1)} + \int_1^{r_m/r_0} d\tilde{r}\,\tilde{r}^{d-1}\left[\frac{\tilde{t}_F(\tilde{r})}{l} - \frac{1}{d}\frac{1}{\tilde{r}}\right]\right\} \qquad r_0 = \frac{2\pi l^2}{d\Delta\phi}\left(1 + \sqrt{1 - \frac{\Phi^2}{\Phi_{max}^2}}\right)$$



Sketch of WDW

Evaluation for CV 2.0 : Complexity=Spacetime volume

CV2.0 complexity of formation density

$$\mathcal{C}_2 = \frac{2}{G}\left(\frac{r_0}{l}\right)^{d-1}\left\{-\frac{1}{d(d-1)} + \int_1^{r_m/r_0} d\tilde{r}\,\tilde{r}^{d-1}\left[\frac{\tilde{t}_F(\tilde{r})}{l} - \frac{1}{d}\frac{1}{\tilde{r}}\right]\right\}$$

$$r_0 = \frac{2\pi l^2}{d\Delta\phi}\left(1+\sqrt{1-\frac{\Phi^2}{\Phi^2_{max}}}\right)$$

- Scaling $\mathcal{C}_2 \propto \Delta\phi^{-(d-1)}$
- Phase transition

➢ AdS soliton / periodic AdS

➢ Order parameter

non-zero in AdS soliton

zero in periodic AdS!

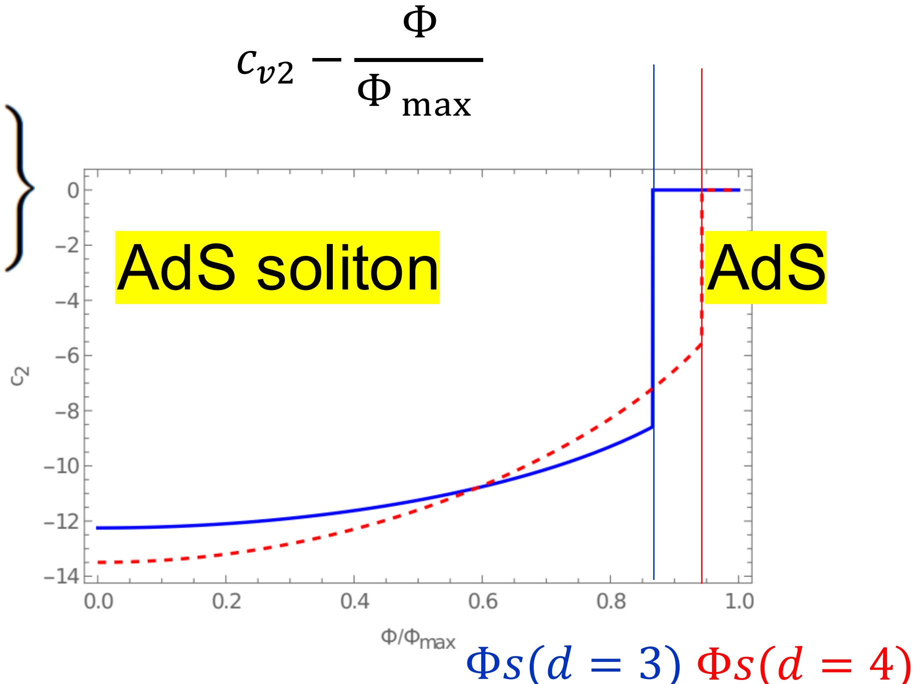
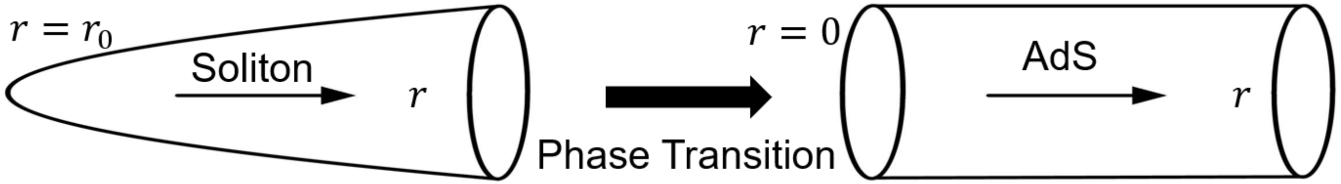
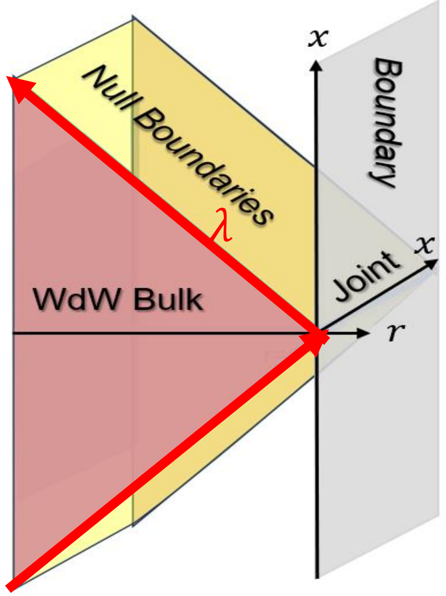


Figure 3. CV2.0 complexity of formation per boundary volume for $d=3$ (solid blue) and $d=4$ (dashed red). Plotted curves are $c_2 \equiv G\mathcal{C}_2(\Delta\phi/l)^{d-1}$, so complexity scales with an inverse power of $\Delta\phi$



Evaluation for CA : Complexity=Action



- λ is the null parameter
- $k_F, k_P \equiv dx/d\lambda$ are the future/past null vectors
- γ_{ij} is the induced metric at fixed λ
- Θ_F, Θ_P are the expansions
- β is an arbitrary parameter
- We can also add a Maxwell boundary term by setting the F^2 prefactor to $(2\nu - 1)/4$.

$$C_A = \frac{S}{\pi\hbar}$$

$$S = S_{bulk} + S_{bdy} + S_{joint}$$

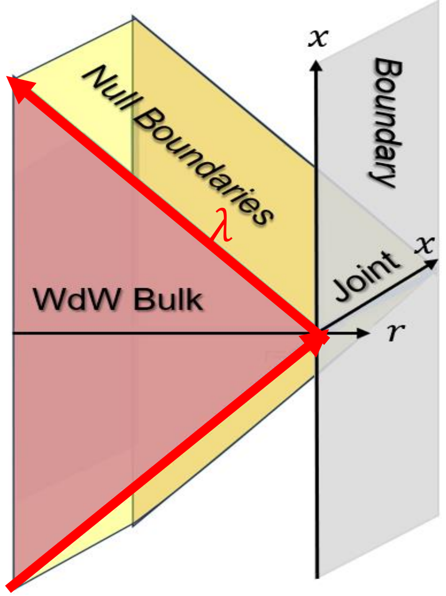
$$S_{\rm bulk} = \frac{1}{16\pi G}\int_{\rm WDW} d^{d+1}x\sqrt{-g}\left(R + \frac{d(d-1)}{l^2} - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right),$$

$$S_{\rm bdy} = -\frac{1}{8\pi G}\int_F d\lambda\, d^{d-2}\vec{x}\, d\phi\sqrt{\gamma}\left(\kappa_F + \Theta_F \ln|\beta l \Theta_F|\right) + \frac{1}{8\pi G}\int_P d\lambda\, d^{d-2}\vec{x}\, d\phi\sqrt{\gamma}\left(\kappa_P + \Theta_P \ln|\beta l \Theta_P|\right),$$

$$S_{\rm joint} = -\frac{1}{8\pi G}\int_{F\cap P} d^{d-2}\vec{x}\, d\phi\sqrt{\gamma}\ln|k_F \cdot k_P/2|\,.$$

arXiv:1609.00207 (Lehner, Myers, Poisson, Sorkin 2016)

Evaluation for CA : Complexity=Action



$$S_{\text{bulk}} = \frac{1}{16\pi G}\int_{\text{WDW}} d^{d+1}x\sqrt{-g}\left(R + \frac{d(d-1)}{l^2} - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right),$$

$$S_{\text{bdy}} = -\frac{1}{8\pi G}\int_F d\lambda\, d^{d-2}\vec{x}\, d\phi\sqrt{\gamma}\left(\kappa_F + \Theta_F \ln|\beta l \Theta_F|\right)$$

$$+\frac{1}{8\pi G}\int_P d\lambda\, d^{d-2}\vec{x}\, d\phi\sqrt{\gamma}\left(\kappa_P + \Theta_P \ln|\beta l \Theta_P|\right),$$

$$S_{\text{joint}} = -\frac{1}{8\pi G}\int_{F\cap P} d^{d-2}\vec{x}\, d\phi\sqrt{\gamma}\ln|k_F\cdot k_P/2|\,.$$

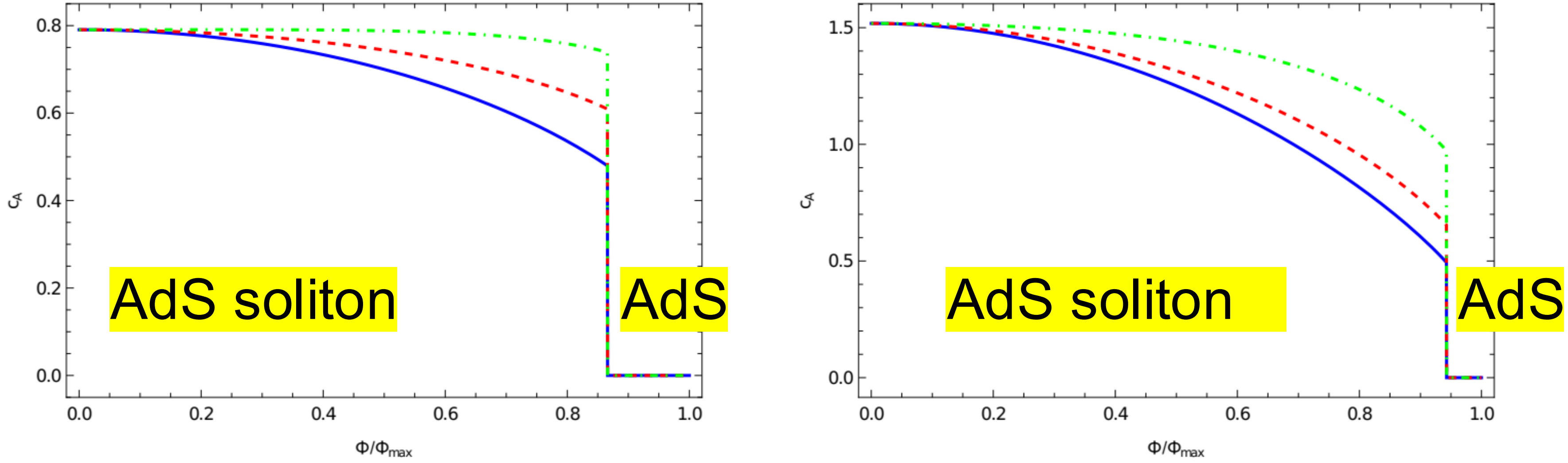
$$C_A = \frac{S}{\pi\hbar}$$

$$S = S_{\text{bulk}} + S_{\text{bdy}} + S_{\text{joint}} - S_{\text{AdS}} = \frac{V_{\vec{x}}\Delta\phi}{8\pi G}\left(\frac{r_0}{l}\right)^{d-1}\left[-\ln(d-1) + I(r_m/r_0, \Phi)\right]$$

arXiv:1609.00207 (Lehner, Myers, Poisson, Sorkin 2016)

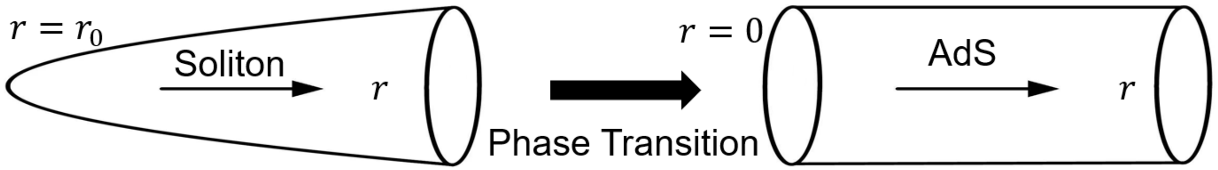
Evaluation for CA: Complexity=Action

- Scaling $\mathcal{C}_A \propto \Delta\phi^{-(d-1)}$



(a) $d = 3$. (b) $d = 4$.

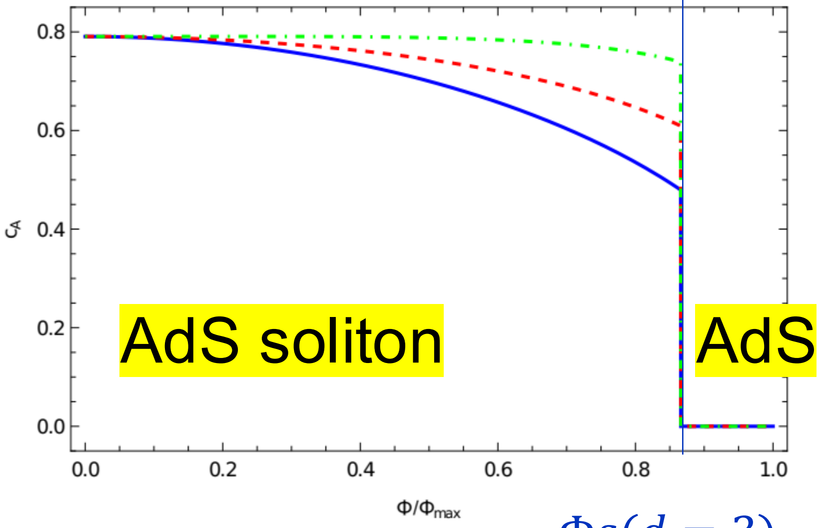
Figure 4. CA complexity of formation per unit boundary volume for $d = 3, 4$ as labeled with $\nu = 0$ (solid blue), $\nu = 1/(d-1)$ (dashed red), and $\nu = 1$ (dot-dashed green). Plotted curves are $c_A = G\mathcal{C}_A(\Delta\phi/l)^{d-1}$ as a function of the Wilson line Φ.



Evaluation for CA: Complexity=Action

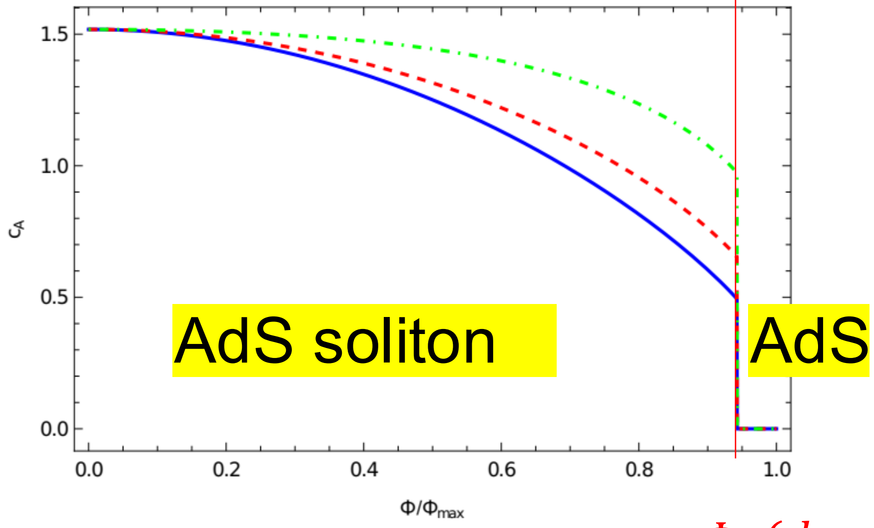
- Phase transition: AdS soliton / periodic AdS, order parameter

$$c_A - \frac{\Phi}{\Phi_{max}}$$



$\Phi s(d=3)$

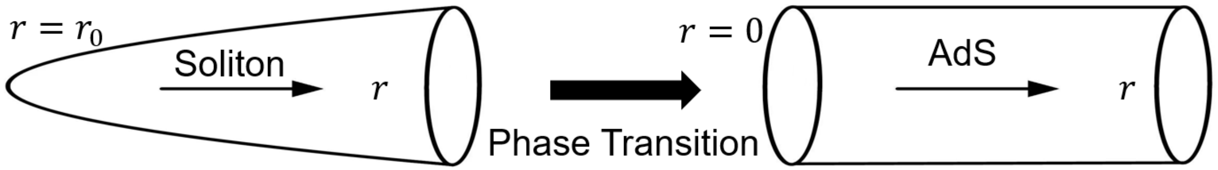
(a) $d = 3$.



$\Phi s(d=4)$

(b) $d = 4$.

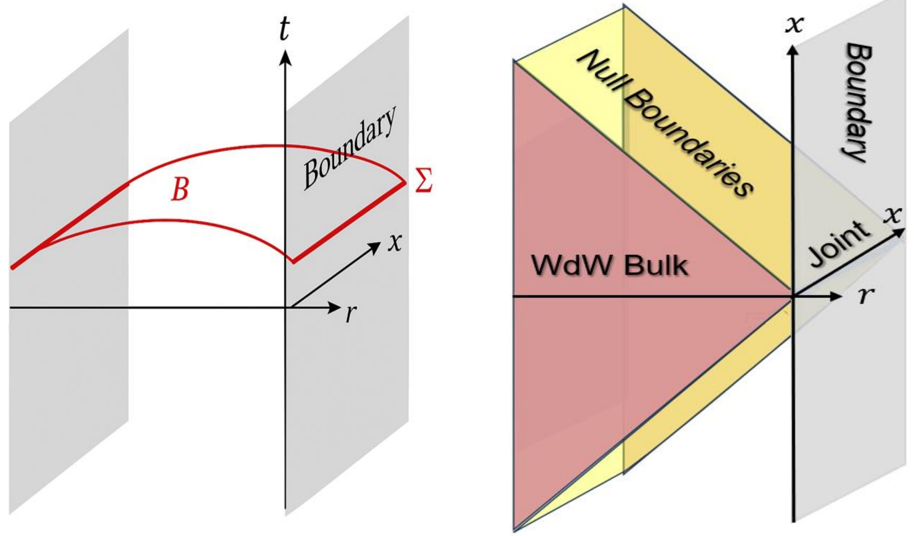
Figure 4. CA complexity of formation per unit boundary volume for $d = 3, 4$ as labeled with $\nu = 0$ (solid blue), $\nu = 1/(d-1)$ (dashed red), and $\nu = 1$ (dot-dashed green). Plotted curves are $c_A = G\mathcal{C}_A(\Delta\phi/l)^{d-1}$ as a function of the Wilson line Φ.



Complexity, scaling, and a phase transition

Jiayue Yang, and Andrew R. Frey (JHEP09(2023)029)

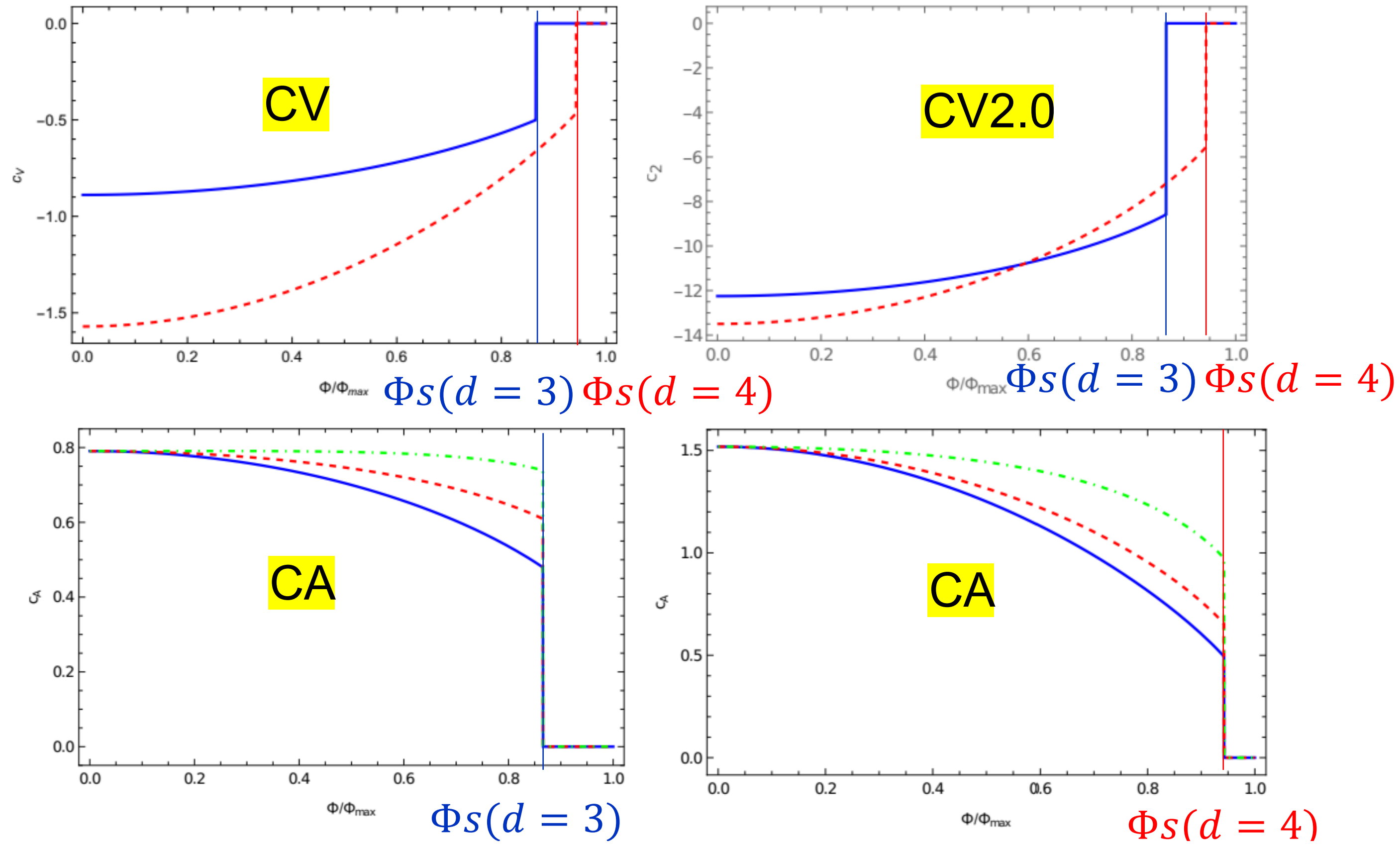
- Holographic complexity

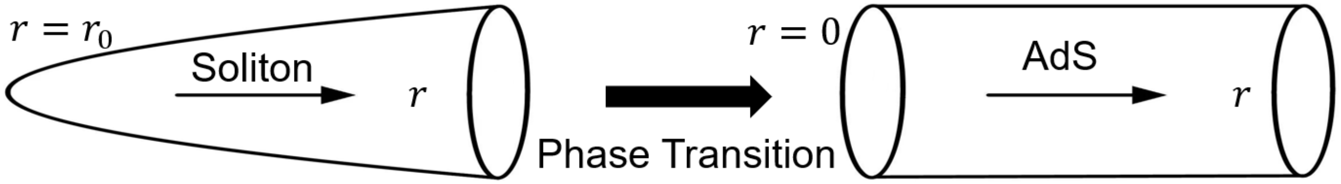


- Same scaling behavior

$$\mathcal{C} \propto \Delta\phi^{-(d-1)}$$

- Same phase transition





CONCLUSION

Conclusion

We evaluated various proposals for holographic complexity in a d=3 and d=4 CFT on a circle (times Minkowski spacetime) with a U(1) gauge field.

- Main results: scaling behavior

The complexity of formation density scales as $\frac{1}{\Delta\phi^{d-1}}$

“Complexity=anything" program also exhibits the same scaling.

- Main results: phase transition

We observe magnetized AdS soliton / periodic AdS phase transition.

The complexity of formation density acts as an order parameter.

Acknowledgments

- JY would like to thank Robert Mann, Niayesh Afshordi, Haijun Wang, and Wencong Gan for encouragement in pursuing this project.
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- The work of JY was supported by the Mitacs Globalink program.

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Thank you for your attention!

I also want to thank the friends who have supported me!

APPENDIX

1

Qubit

Concept / Definition 1.5 (Computational Basis States/ Basis Vectors)

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \equiv e_0 \tag{1.4}$$

$$|1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \equiv e_1 \tag{1.5}$$

♣

Concept / Definition 1.6 (Qubits/ Quantum digital)

There are infinite explicit states of a qubit. If we measure the qubit, the state is $|0\rangle$ *with probability* $|\alpha_0|^2$*; the state is* $|1\rangle$ *with probability* $|\alpha_1|^2$*. We can use the probability amplitudes vector to label the state of qubit, which is the superposition of* $|0\rangle$ *and* $|1\rangle$

Set probability vector :

$$\begin{bmatrix} \alpha_0 \\ \alpha_1 \end{bmatrix} = \begin{bmatrix} \sin\theta \\ e^{i\theta}\cos\theta \end{bmatrix} = \alpha_0 |0\rangle + \alpha_1 |1\rangle \tag{1.6}$$

where the probability amplitudes $\alpha_0, \alpha_1 \in \mathbb{C}$, $|\alpha_0|^2 + |\alpha_1|^2 = 1$

♣

Quantum gate

- a **quantum logic gate** (or simply **quantum gate**) is a basic quantum circuit operating on a small number of qubits.

- Quantum logic gates are the building blocks of quantum circuits, like classical logic gates are for conventional digital circuits.

Example 1.3 Pauli matrices

Bit flip

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{1.15}$$

$$\sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \tag{1.16}$$

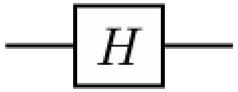
Phase flip

$$\sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{1.17}$$

■

The Pauli-*X* gate is the quantum equivalent of the NOT gate for classical computers with respect to the standard basis $|0\rangle$, $|1\rangle$, which distinguishes the *z* axis on the Bloch sphere. It is sometimes called a bit-flip as it maps $|0\rangle$ to $|1\rangle$ and $|1\rangle$ to $|0\rangle$. Similarly, the Pauli-*Y* maps $|0\rangle$ to $i|1\rangle$ and $|1\rangle$ to $-i|0\rangle$. Pauli *Z* leaves the basis state $|0\rangle$ unchanged and maps $|1\rangle$ to $-|1\rangle$. Due to this nature, Pauli *Z* is sometimes called phase-flip.

Example 1.2 Reflection

Hadamard gate 

$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \tag{1.9}$$

$$H|0\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = |+\rangle \tag{1.10}$$

$$H|+\rangle = \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 2 \\ 0 \end{bmatrix} = |0\rangle \tag{1.11}$$

$$H|1\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) = |-\rangle \tag{1.12}$$

$$H|-\rangle = \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 0 \\ 2 \end{bmatrix} = |1\rangle \tag{1.13}$$

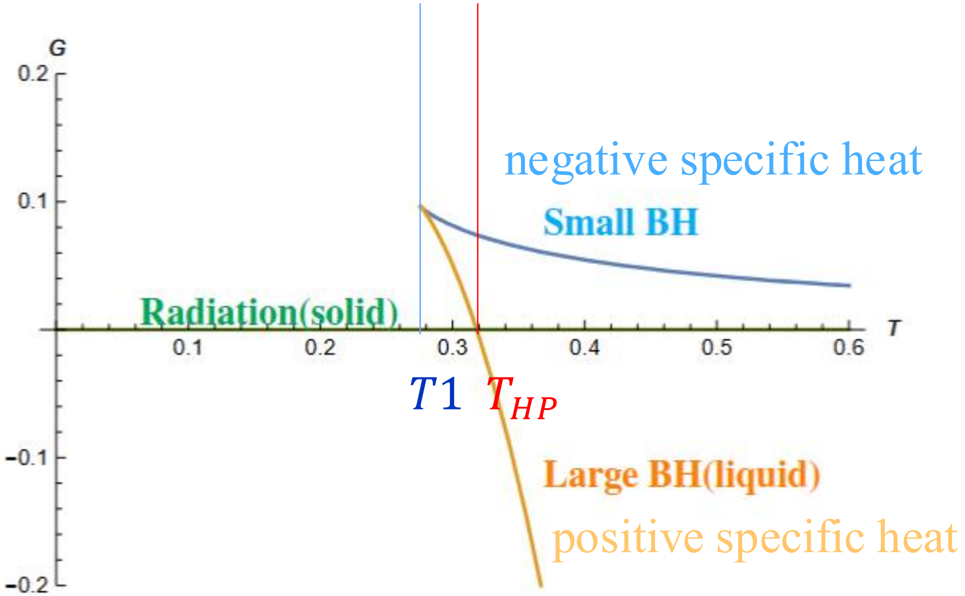
$$|0\rangle \xleftrightarrow{H} |+\rangle\,; \qquad |1\rangle \xleftrightarrow{H} |-\rangle \tag{1.14}$$

It is possible to guess the slope of the curve during the linear growth. The guess is based on two observations. The first is the complexity is extensive; it is proportional to the number of active degrees of freedom of the system. We may use the thermal entropy as a measure of the size of the system. Therefore complexity and its rate of growth should be proportional to the entropy of the system.

The second observation is that the slope is a rate and should have units of inverse time or energy. One possibility is that the rate of growth is proportional to the total energy of the system, but this can't be right. The ground states of systems such as extremal black holes have both entropy and energy but obviously whatever complexity they may have is time independent. The right quantity is the product of the entropy and the temperature.

$$\frac{d\mathcal{C}}{dt} = TS \tag{1.7}$$

Example



first-order phase transition

Figure 3.1: Hawking–Page transition (Blue curve is small BH, orange curve is large BH, Green curve is radiation) at fixed $P = \frac{3}{8\pi}$

1. $T \leq T_1$(the intersection of Small BH and Large BH), no black hole can exist. It is just the thermal AdS spacetime.
2. $T_1 \leq T \leq T_{HP}$ Black holes are unstable and will emit Hawking radiation to gradually change to thermal AdS spacetime.
3. $T \geq T_{HP}$ Preferred phase are large black holes.There is a first-order phase transition between thermal radiation and large black holes.

Example

3.1.1 Hawking–Page transition between the thermal radiation and large black holes

Uncharged Black hole solution(dim=3+1)

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2 \tag{3.3}$$

where

$$f(r) = 1 - \frac{2M}{r} + \frac{r^2}{l^2} \tag{3.4}$$

$$f(r) = k - \frac{2M}{r} + \frac{r^2}{l^2} \tag{3.5}$$

with $k = 1$ being the (d − 2)-sphere(spherical horizon geometry), $k = 0$ being a torus(planar horizon geometry), and $k = 1$ being a compact hyperbolic space (hyperbolic horizon geometry)

Set $f(r) = 0$, we find the largest zero of $f(r)$, denoted it by r_+ So we have

$$f(r_+) = k - \frac{2M}{r_+} + \frac{r_+^2}{l^2} = 0 \tag{3.6}$$

Solve this, we get

$$M = \frac{r_+}{2}(k + \frac{r_+^2}{l^2}) \tag{3.7}$$

Actually

$$M = \frac{A_k r_+}{8}(k + \frac{r_+^2}{l^2}) \tag{3.8}$$

where $A_{k=1} = 4$ for the spherical case

The entropy S is

$$S = \frac{A}{4} = \frac{\pi r_+^2 A_k}{4} \tag{3.9}$$

The pressure P is given by

$$P = \frac{(d-1)(d-2)}{16\pi l^2} = \frac{3}{8\pi l^2} \tag{3.10}$$

The volume is

$$V \equiv \left(\frac{\partial M}{\partial P}\right)_{S,Q,J} = \frac{\partial}{\partial P}\left(\frac{A_k r_+}{8}(k + \frac{8\pi P r_+^2}{3})\right) = \frac{A_k \pi r_+^3}{3} \tag{3.11}$$

Hence the temperature is

$$\begin{aligned}
T &= \frac{f'(r_+)}{4\pi} = \frac{1}{4\pi}(\frac{2M}{r^2} + \frac{2r}{l^2})|r = r_+ && (3.12)\\
&= \frac{1}{4\pi}(\frac{2Ml^2 + 2r_+^3}{l^2 r_+^2}) && (3.13)\\
&= \frac{1}{4\pi}(\frac{\frac{A_k}{4}(r_+ k l^2 + r_+^3) + 2r_+^3}{l^2 r_+^2}) && (3.14)\\
&= \frac{\frac{A_k}{4} k l^2 + (\frac{A_k}{4} + 2) r_+^2}{4\pi l^2 r_+} && (3.15)\\
& && (3.16)
\end{aligned}$$

To be specific, we consider the $A_k = 4(k = 1)$ spherical case,

$$M = \frac{r_+}{2}(k + \frac{r_+^2}{l^2}), \quad S = \pi r_+^2, \quad T = \frac{kl^2 + 3r_+^2}{4\pi l^2 r_+}, \quad V = \frac{4\pi r_+^3}{3} \tag{3.17}$$

$$M = \frac{r_+}{2}(1 + \frac{r_+^2}{l^2}), \quad S = \pi r_+^2, \quad T = \frac{l^2 + 3r_+^2}{4\pi l^2 r_+}, \quad P = \frac{3}{8\pi l^2}, \quad V = \frac{4\pi r_+^3}{3} \tag{3.18}$$

Gibbs free energy

$$\begin{aligned}
G &= M - TS && (3.19)\\
&= \frac{r_+}{2} + \frac{r_+^3}{2l^2} - \frac{r_+}{4}(1 + 3\frac{r_+^2}{l^2}) && (3.20)\\
&= \frac{2l^2 r_+^2 + 2r_+^4 - l^2 r_+^2 - 3r_+^4}{4l^2 r_+} && (3.21)\\
&= \frac{l^2 r_+^2 - r_+^4}{4l^2 r_+} && (3.22)\\
&= \frac{l^2 r_+ - r_+^3}{4l^2} && (3.23)\\
& && (3.24)
\end{aligned}$$

For the expression of temperature T (3.18), solve for r_+, we get the radius for large BH

$$r_+ = \frac{1}{6}(4\pi T + \sqrt{-12 + 16\pi^2 T^2}) \tag{3.25}$$

the radius for small BH

$$r_+ = \frac{1}{6}(4\pi T - \sqrt{-12 + 16\pi^2 T^2}) \tag{3.26}$$

2

MORE ABOUT HOLOGRAPHIC COMPLEXITY

More about holographic complexity

What boundary state are we considering?

- Well, we're looking at the ground/vacuum state with those parameters (compactification radius, Wilson line value).
- But the ground state changes depending on the Wilson line.
- At small values, the ground state is confining (like QCD).
- It becomes deconfined (like perturbative particle physics) at large Wilson line.

Negativity for CV and Positivity for CA?

- We should be careful.
- In every case, whether it is for the soliton or for periodic AdS, complexity is the "distance" as measured from some reference state of the QFT system. But we don't know what that reference state is.
- Note that even the complexity of periodic AdS diverges if you let the cut off radius get large. Complexity of formation does not say that we choose the periodic AdS as the reference state. Instead it says that we just take the difference of the complexity of the soliton and periodic AdS.
- That is why the complexity of formation can be negative. In CV and CV2.0 cases, the soliton is actually closer to the reference state than the periodic AdS is.

Negativity for CV and Positivity for CA?

- We conjecture that the reference states for the CV and CV2.0 proposals lie in a different region of the CFT's Hilbert space than the reference states for CA complexity proposals (which may vary with varying boundary conditions for the gauge field).

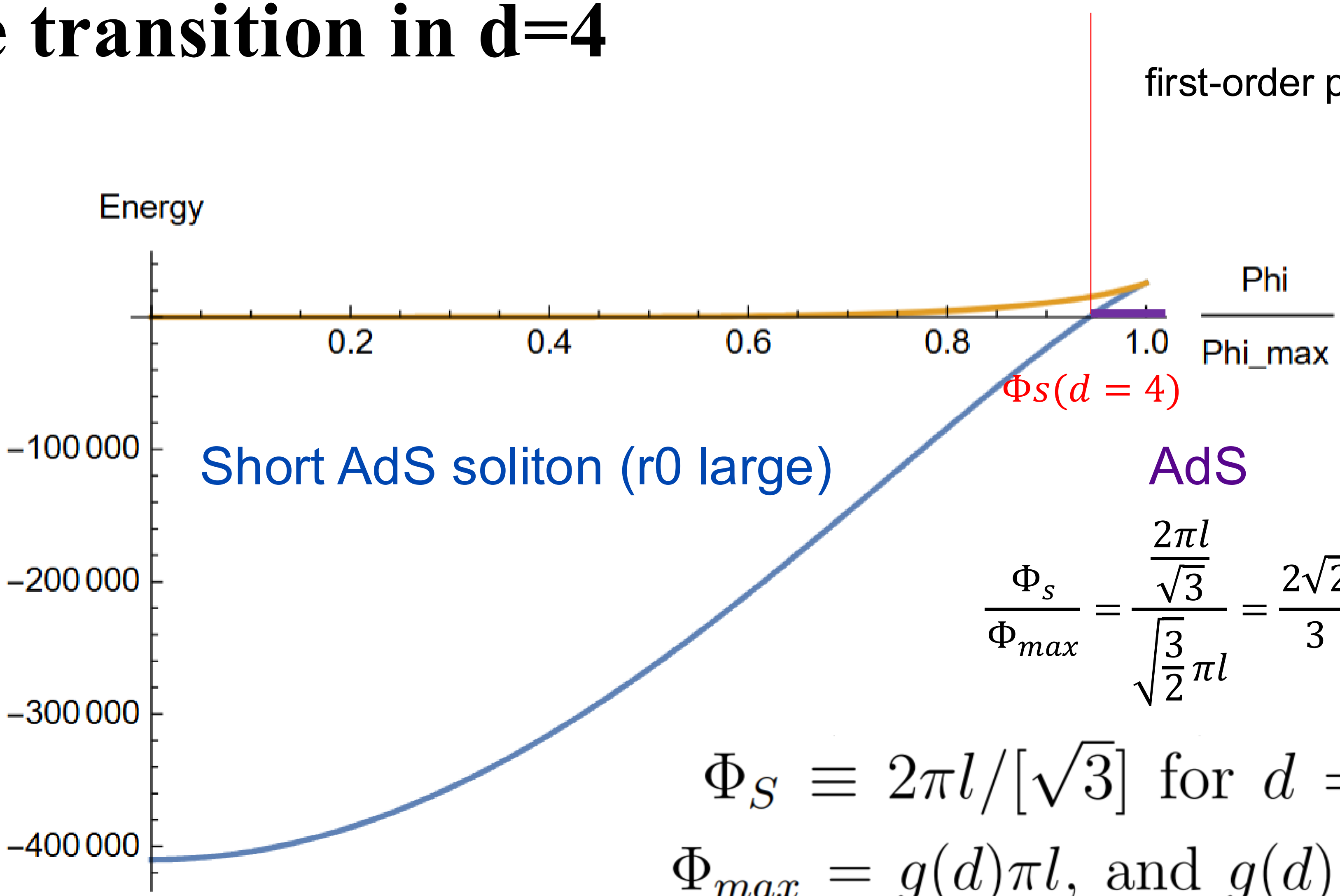
- Another interpretation is that the volume and action complexity proposals have similar reference states but very different distance measures,

- though that seems less likely to lead to the observed sign change

MORE ABOUT PHASE TRANSITION

More about phase transition

Phase transition in d=4



$$\frac{\Phi_s}{\Phi_{max}} = \frac{\frac{2\pi l}{\sqrt{3}}}{\sqrt{\frac{3}{2}}\pi l} = \frac{2\sqrt{2}}{3} = 0.9428$$

$$\Phi_S \equiv 2\pi l/[\sqrt{3}] \text{ for } d = 3 \ [d = 4]$$

$$\Phi_{max} = g(d)\pi l, \text{ and } g(d) \equiv \sqrt{(46 - 10d)/d}$$

Figure 4: Phase transition of the generalized soliton

Order parameter

ferromagnetic /paramagnetic phase transition

Ordered magnetic moments (ferromagnetic, Figure 1) below the Curie temperature.
$M(T) \neq 0$

The order parameter is the magnetization $M(T)$ that goes from a finite value to zero when the temperature is increased above the Curie temperature.

Disordered magnetic moments (paramagnetic, Figure 2) above the Curie temperature.
$M(T) = 0$

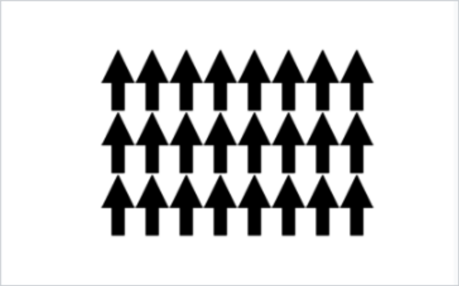


Figure 1. Below the Curie temperature, neighbouring magnetic spins align parallel to each other in a ferromagnet in the absence of an applied magnetic field.

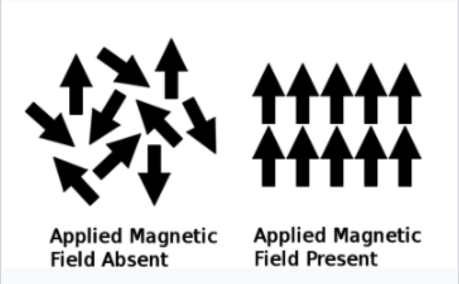


Figure 2. Above the Curie temperature, the magnetic spins are randomly aligned in a paramagnet unless a magnetic field is applied.

https://en.wikipedia.org/wiki/Curie_temperature

SOME DERIVATIONS

Some derivations

Derive Energy expression

There are two approaches to determine the energy density of each solution and thereby identify the ground state of the CFT for a given periodicity and Wilson line. The first approach is holographic renormalization [88, 89, 90], and the second approach is the extrinsic curvature method [91]. We will first discuss the holographic renormalization method. Here, we perform a Fefferman-Graham expansion [88] of the asymptotic form of the bulk metric as follows:

$$ds^2 = l^2 \frac{dz^2}{z^2} + \frac{l^2}{z^2}\left(\gamma_{ij}^{(0)} + \sum_{n=d}^{\infty} z^n \gamma_{ij}^{(n)}\right) dx^i dx^j; \tag{4.7}$$

with $x^i \equiv [t, \vec{x}, \phi]$ in our case, and the conformal boundary is located at $z = 0$. In the expansion, $\gamma_{ij}^{(0)}$ is related to the boundary CFT background metric, and $\gamma_{ij}^{(d)}$ is used to compute the expectation value of the CFT energy-momentum tensor. In general, the sum can include terms with $1 \leq n < d$, as well as terms proportional to $z^n \ln(z)$ for $n \geq d$ when d is even. However, these latter terms vanish for the solitons we consider. Apparently

$$\int \frac{dz}{z} = -\int \frac{dr}{r\sqrt{f(r)}} \quad \Rightarrow \quad \ln(z/l) = -\ln(r/l) + \frac{\mu l^2}{2dr^d} + \cdots \tag{4.8}$$

using a large radius expansion. Solving iteratively, we obtain

$$r = \frac{l^2}{z}\left(1 + \frac{\mu l^2}{2d}\frac{z^d}{l^{2d}} + \cdots\right). \tag{4.9}$$

In holographic renormalization, as discussed in references such as [89, 90], the expectation value of boundary stress-energy tensor is

$$\langle T_{ij} \rangle = \frac{d l^{d-1} \gamma_{ij}^{(d)}}{16\pi G} + \cdots, \tag{4.10}$$

Derive Energy expression

Derive (4.8) (4.9)

solutions are generalizations of the AdS soliton, and both have the following metric (fo
$d = 3, 4$)

$$ds^2 = \frac{r^2}{l^2}\left(-dt^2 + d\vec{x}^2 + f(r)d\phi^2\right) + \frac{l^2}{r^2 f(r)}dr^2, \quad f(r) \equiv 1 - \frac{\mu l^2}{r^d} - \frac{Q^2 l^2}{r^{2d-2}}, \qquad (4.2$$

Here, we perform a Fefferman-Graham expansion [88] of the asymptotic form of the bul
metric as follows:

$$ds^2 = l^2 \frac{dz^2}{z^2} + \frac{l^2}{z^2}\left(\gamma_{ij}^{(0)} + \sum_{n=d}^{\infty} z^n \gamma_{ij}^{(n)}\right) dx^i dx^j; \qquad (4.$$

with $x^i \equiv [t, \vec{x}, \phi]$ in our case, and the conformal boundary is located at $z = 0$. In the expansion, $\gamma_{ij}^{(0)}$ is related to the boundary CFT background metric, and $\gamma_{ij}^{(d)}$ is used to compute the expectation value of the CFT energy-momentum tensor. In general, the sum can include terms with $1 \leq n < d$, as well as terms proportional to $z^n \ln(z)$ for $n \geq d$ when d is even. However, these latter terms vanish for the solitons we consider. Apparently

$$\int \frac{dz}{z} = -\int \frac{dr}{r\sqrt{f(r)}} \quad \Rightarrow \quad \ln(z/l) = -\ln(r/l) + \frac{\mu l^2}{2dr^d} + \cdots \qquad (4.8)$$

using a large radius expansion. Solving iteratively, we obtain

$$r = \frac{l^2}{z}\left(1 + \frac{\mu l^2}{2d}\frac{z^d}{l^{2d}} + \cdots\right). \qquad (4.9)$$

?

(4.8) radial coordinate

$$l^2\frac{dz^2}{z^2} = \frac{l^2}{r^2 f(r)}dr^2 \Rightarrow \int\frac{dz}{z} = -\int\frac{1}{r\sqrt{f}}dr$$

$$\frac{1}{\sqrt{f}} = \frac{1}{\sqrt{1-\frac{\mu l^2}{r^d}-\frac{Q^2 l^2}{r^{2d-2}}}} \xrightarrow{r\to\infty} 1+\frac{\mu l^2}{2r^d}$$

$$\frac{dz}{z} = -\frac{dr}{r}\left(1+\frac{\mu l^2}{2r^d}\right)$$

$$\frac{d(\frac{z}{l})}{(\frac{z}{l})} = -\frac{d\frac{r}{l}}{\frac{r}{l}}\left(1+\frac{\mu l^{2-d}}{2(\frac{r}{l})^d}\right) = -\frac{d\frac{r}{l}}{\frac{r}{l}} - d\frac{r}{l}\,\frac{\mu l^{2-d}}{2(\frac{r}{l})^{d+1}}$$

$$\ln\left(\frac{z}{l}\right) = -\ln\left(\frac{r}{l}\right) + \frac{\mu l^{2-d}}{d\,2\,(\frac{r}{l})^d}$$

(4.9)

$$\ln\left(\frac{z}{l}\right) = -\ln\left(\frac{r}{l}\right) + \frac{\mu l^2}{2dr^d}$$

$$\frac{\mu l^2}{2dr^d} = \ln\left(\frac{z}{l}\frac{r}{l}\right)$$

$$e^{\frac{\mu l^2}{2dr^d}} = \frac{zr}{l^2} \Rightarrow r = \frac{l^2}{z}\left(1+\frac{\mu l^2}{2dr^d}+\cdots\right)$$

leading order $r = \frac{l^2}{z}$

$$r = \frac{l^2}{z}\left(1+\frac{\mu l^2}{2d}\frac{z^d}{l^{2d}}\right)$$

Derive Energy expression

Derive (4.14) (4.15)

solutions are generalizations of the AdS soliton, and both have the following metric (for $d = 3, 4$)

$$ds^2 = \frac{r^2}{l^2}\left(-dt^2 + d\vec{x}^2 + f(r)d\phi^2\right) + \frac{l^2}{r^2 f(r)}dr^2, \quad f(r) \equiv 1 - \frac{\mu l^2}{r^d} - \frac{Q^2 l^2}{r^{2d-2}}, \tag{4.2}$$

$d-2$ 维

For the soliton backgrounds with a surface at $r = r_m$, the lapse function is $N = r_m/l$, the radial unit vector has only one nontrivial component $n^r = r_m\sqrt{f(r_m)}/l$, and the area of the surface in the limit $r_m \to \infty$ is given by

$$A = V_{\vec{x}}\Delta\phi\left(\frac{r_m}{l}\right)^{d-1}\sqrt{f(r_m)}, \tag{4.13}$$

with $V_{\vec{x}}$ denotes the volume along the $\vec{x}$ directions. Thus, we obtain

$$\int NK = \frac{V_{\vec{x}}\Delta\phi}{l^{d+1}}\left[(d-1)r_m^d + \left(1 - \frac{d}{2}\right)\mu l^2\right]. \tag{4.14}$$

And NK_0 has the same form with $\mu = 0$, except that we must change the asymptotic periodicity by taking $\Delta\phi \to \sqrt{f(r_m)}\Delta\phi$, ensuring that the two surfaces have the same proper size at the cutoff radius r_m, as emphasized in [78]. Finally, we obtain

$$E = -\frac{V_{\vec{x}}\Delta\phi\mu}{16\pi G l^{d-1}}, \tag{4.15}$$

which is consistent with the holographic stress-energy tensor. Instead of matching the proper size of the cutoff surfaces in the soliton and periodic AdS spacetimes, we could alternatively take cutoff surfaces at the same Fefferman-Graham coordinate z, as will be

$$f' = d\, r^{-d-1}\mu l^2 + (2d-2)\, r^{-2d+1} Q^2 l^2$$

$$\sqrt{f(r_m)}\Delta\phi = \left(1 - \frac{1}{2}\frac{\mu l^2}{r^d}\right)\Delta\phi$$

(4.14)

$$\int NK = N n^\mu \partial_\mu A$$
$$= \frac{r_m}{l}\frac{r_m\sqrt{f(r_m)}}{l} V_{\vec{x}}\Delta\phi\left((d-1)\frac{r_m^{d-2}}{l^{d-1}}\sqrt{f(r_m)} + \left(\frac{r_m}{l}\right)^{d-1}\frac{f'(r_m)}{2\sqrt{f(r_m)}}\right)$$
$$= \frac{V_{\vec{x}}\Delta\phi}{l^{d+1}}\left((d-1)r_m^d\,(f(r_m)) + r_m^{d+1}\frac{f'(r_m)}{2}\right)$$
$$= \frac{V_{\vec{x}}\Delta\phi}{l^{d+1}}\left((d-1)r_m^d\left(1 - \frac{\mu l^2}{r_m^d}\right) + r_m^{d+1}\frac{d\, r_m^{-d-1}\mu l^2}{2}\right)$$
$$= \frac{V_{\vec{x}}\Delta\phi}{l^{d+1}}\left((d-1)r_m^d + (1-d)\mu l^2 + \frac{d}{2}\mu l^2\right)$$
$$= \frac{V_{\vec{x}}\Delta\phi}{l^{d+1}}\left((d-1)r_m^d + \left(1-\frac{d}{2}\right)\mu l^2\right)$$

(4.15)

$$E = -\frac{1}{8\pi G}\int N(K - K_0)$$
$$= -\frac{1}{8\pi G}\left[\frac{V_{\vec{x}}\Delta\phi}{l^{d+1}}\left((d-1)r_m^d + \left(1-\frac{d}{2}\right)\mu l^2\right) - \frac{V_{\vec{x}}}{l^{d+1}}\left(1 - \frac{1}{2}\frac{\mu l^2}{r_m^d}\right)\Delta\phi\,(d-1)r_m^d\right]$$
$$= -\frac{1}{8\pi G}\left[\frac{V_{\vec{x}}\Delta\phi}{l^{d+1}}\left(\left(1-\frac{d}{2}\right)\mu l^2 + \frac{d-1}{2}\mu l^2\right)\right]$$
$$= -\frac{1}{8\pi G}\left[\frac{V_{\vec{x}}\Delta\phi}{l^{d+1}}\frac{1}{2}\mu l^2\right] = -\frac{1}{16\pi G}\frac{V_{\vec{x}}\Delta\phi\,\mu}{l^{d-1}}$$

The null boundaries of the Wheeler-de Witt patch are given by

$$ds^2 = -\frac{r^2}{l^2}dt^2 + \frac{l^2 dr^2}{r^2 f(r)} = 0 \tag{205}$$

$$-\frac{r^2}{l^2}\frac{dt^2}{dr^2} + \frac{l^2}{r^2 f(r)} = 0 \tag{206}$$

$$\frac{dt}{dr} = \frac{l^2}{r^2\sqrt{f(r)}} \tag{207}$$

$$t = \pm l^2 \int_r^{r_{max}} \frac{dr'}{r'^2\sqrt{f(r')}} \tag{208}$$

$$\tag{209}$$

If we set $r = r_+\tilde{r}$, $t = \tilde{t}r_+$, notice that $r_0 = r_+$ Then

$$\tilde{t} = \pm l^2 \int_{\tilde{r}}^{r_{max}/r_+} \frac{dr'}{r'^2\sqrt{f(r')}} \tag{210}$$

CONFINEMENT

Confinement

Quark confinement

1. Originally, due to **quark confinement**, an infinite amount of energy is required to isolate a single quark, so there are no free quarks.
2. However, at sufficiently high temperatures, a **quark-gluon plasma** can form, in which quarks can move freely to some extent. This is a **deconfining phase**.
3. The process of heating corresponds to the confinement/deconfinement phase transition.

Gravity dual of confinement

1. In the gravitational duality, physicists map the potential between the quark-antiquark to a string in the bulk.
2. The two ends of the string are on the boundary, corresponding to the quark and the antiquark.
3. If there is **no horizon(soliton),** the string just stretches along the normal geometry at the minimum radius. The string's energy is proportional to the distance between the ends. To separate the ends infinitely far apart requires infinite energy, corresponding to the **confining phase**.
4. If there's **a horizon(Poincare AdS),** the string can drop down arbitrarily close to the horizon as the ends move apart from each other. Because of the infinite gravitational redshift, that costs essentially zero energy, **deconfining phase**.

Soliton no horizon, Poincare AdS horizon

1. The AdS soliton doesn't have a horizon at r_0. (r_0 is just an origin in the $r-\phi$ plane). You can tell it's not a horizon because the metric component $g_{tt} \neq 0$ there. So all the AdS solitons (charged and uncharged) are confining.

$$ds^2 = \frac{r^2}{l^2}\left(-dt^2 + d\vec{x}^2 + f(r)d\phi^2\right) + \frac{l^2}{r^2 f(r)}dr^2, \qquad f(r) \equiv 1 - \frac{\mu l^2}{r^d} - \frac{Q^2 l^2}{r^{2d-2}}$$

2. We are working in Poincare coordinates, where AdS looks like a radius and Minkowski spacetime (in our case with a periodic direction), so the boundary is Minkowski. Even empty/vacuum AdS has a horizon at r=0 in Poincare coordinates.

$$ds^2 = \frac{r^2}{l^2}\left(-dt^2 + d\vec{x}^2 + d\phi^2\right) + \frac{l^2}{r^2}dr^2, \qquad A = -\frac{\Phi}{\Delta\phi}d\phi$$

Hawking-Page transition

1. Hawking-Page transition (thermal AdS/AdS black hole) is also a confinement/deconfinement transition

2. AdS in global coordinates, no horizon (confining)

$$ds^2 = -f(r)\, dt^2 + \frac{1}{f(r)}\, dr^2 + r^2\, d\Omega_{n-2}^2$$

where $f(r) = 1 + \frac{r^2}{\alpha^2}$

1. Black hole has a horizon (deconfining)

first-order phase transition

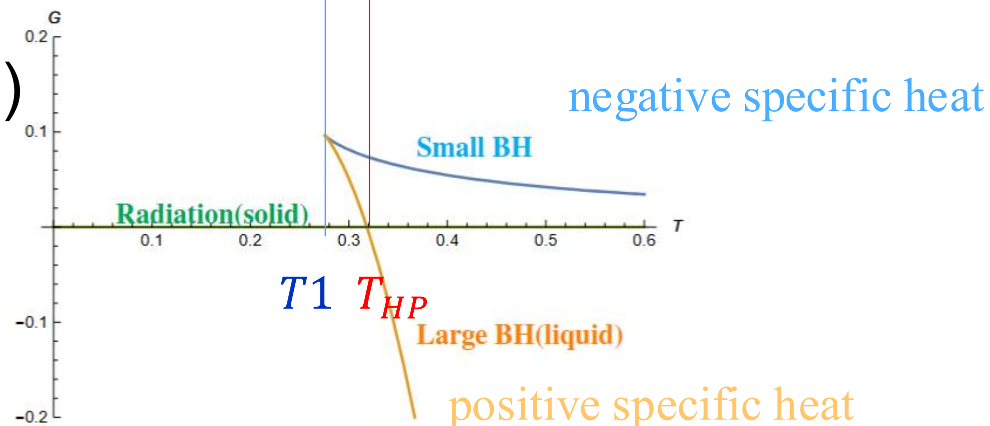


Figure 3.1: Hawking–Page transition (Blue curve is small BH, orange curve is large BH, Green curve is radiation) at fixed $P = \frac{3}{8\pi}$

1. $T \leq T_1$(the intersection of Small BH and Large BH), no black spacetime.
2. $T_1 \leq T \leq T_{HP}$ Black holes are unstable and will emit Hawking radiation to gradually change to thermal AdS spacetime.
3. $T \geq T_{HP}$ Preferred phase are large black holes.There is a first-order phase transition between thermal radiation and large black holes.

Ensemble

- Grand canonical ensemble has fixed μ (chemical potential) V T , so in our case, $\Delta\phi$ is interpreted as the volume (size), potential A_μ(or Wilson line Φ) is like μ (chemical potential).
- Canonical ensemble has fixed N (number of particles) V T, $\Delta\phi$ is still interpreted as volume, and $F_{\mu\nu}$ (or charge Q) is like N (number of particles).
- $\Delta\phi$ is a physical circumference; it's not Euclidean time, so it doesn't relate to temperature. (There are no period in Euclidean time, or you can think has infinite period, zero temperature). At zero temperature, every member of the ensemble must be in the ground state.

Scaling

- Note that the total complexity of formation scales as $V_x \Delta\phi$ (the boundary volume), the bigger the size is, the harder to construct this system.
- What inverse proportional to some power law of $\Delta\phi$ is complexity of formation **density**, the bigger the size is, the average complexity of formation **density** is smaller.
- CFTs do NOT have a scale. For example, no field in a CFT has a mass. It is interesting (and surprising) to find the complexity has a definite power law scaling!
- We have introduced a scale by having the CFT live on a circle of fixed size $\Delta\phi$. And the complexity scaling with this size.

The significance of my work for QG?

- BH are where QT and GT meet, neither the Q effect nor the G effect can be ignored. Only when we consider Q effect, we have BH thermodynamics.
- Holography helps us to understand the relationship between the quantum theory and gravitational theory.
- Originally complexity is defined in a quantum system, and in holography, it is dual to some geometric/gravitational observables. Complexity helps us to understand the interior of BH.

High Energy Physics - Theory

[Submitted on 26 Jul 2017 (v1), last revised 23 Jul 2018 (this version, v2)]

Circuit complexity in quantum field theory

Ro Jefferson, Robert C. Myers

Motivated by recent studies of holographic complexity, we examine the question of circuit complexity in quantum field theory. We provide a quantum circuit model for the preparation of Gaussian states, in particular the ground state, in a free scalar field theory for general dimensions. Applying the geometric approach of Nielsen to this quantum circuit model, the complexity of the state becomes the length of the shortest geodesic in the space of circuits. We compare the complexity of the ground state of the free scalar field to the analogous results from holographic complexity, and find some surprising similarities.

Comments: Corrected typo in eqs. (1.1) and (3.35). Added footnote on pg. 33, and additional references. Author name change

Subjects: **High Energy Physics - Theory (hep-th)**; Quantum Physics (quant-ph)

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High Energy Physics - Theory

[Submitted on 26 Jul 2017 (v1), last revised 13 Nov 2018 (this version, v3)]

Towards Complexity for Quantum Field Theory States

Shira Chapman, Michal P. Heller, Hugo Marrochio, Fernando Pastawski

We investigate notions of complexity of states in continuous quantum-many body systems. We focus on Gaussian states which include ground states of free quantum field theories and their approximations encountered in the context of the continuous version of Multiscale Entanglement Renormalization Ansatz. Our proposal for quantifying state complexity is based on the Fubini-Study metric. It leads to counting the number of applications of each gate (infinitesimal generator) in the transformation, subject to a state-dependent metric. We minimize the defined complexity with respect to momentum preserving quadratic generators which form \mathfrak{su}(1,1) algebras. On the manifold of Gaussian states generated by these operations the Fubini-Study metric factorizes into hyperbolic planes with minimal complexity circuits reducing to known geodesics. Despite working with quantum field theories far outside the regime where Einstein gravity duals exist, we find striking similarities between our results and holographic complexity proposals.

Comments: 6+7 pages, 6 appendices, 2 figures; v2: references added; acknowledgments expanded; appendix F added, reviewing similarities and differences with hep-th/1707.08570; v3: version published in PRL

Subjects: **High Energy Physics - Theory (hep-th)**; Quantum Physics (quant-ph)

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Submission history

High Energy Physics - Theory

[Submitted on 30 Oct 2024]

On the Complexity of Quantum Field Theory

Thomas W. Grimm, Mick van Vliet

We initiate a study of the complexity of quantum field theories (QFTs) by proposing a measure of information contained in a QFT and its observables. We show that from minimal assertions, one is naturally led to measure complexity by two integers, called format and degree, which characterize the information content of the functions and domains required to specify a theory or an observable. The strength of this proposal is that it applies to any physical quantity, and can therefore be used for analyzing complexities within an individual QFT, as well as studying the entire space of QFTs. We discuss the physical interpretation of our approach in the context of perturbation theory, symmetries, and the renormalization group. Key applications include the detection of complexity reductions in observables, for example due to algebraic relations, and understanding the emergence of simplicity when considering limits. The mathematical foundations of our constructions lie in the framework of sharp o-minimality, which ensures that the proposed complexity measure exhibits general properties inferred from consistency and universality.

Comments: 33 pages

Subjects: **High Energy Physics - Theory (hep-th)**; Quantum Physics (quant-ph)

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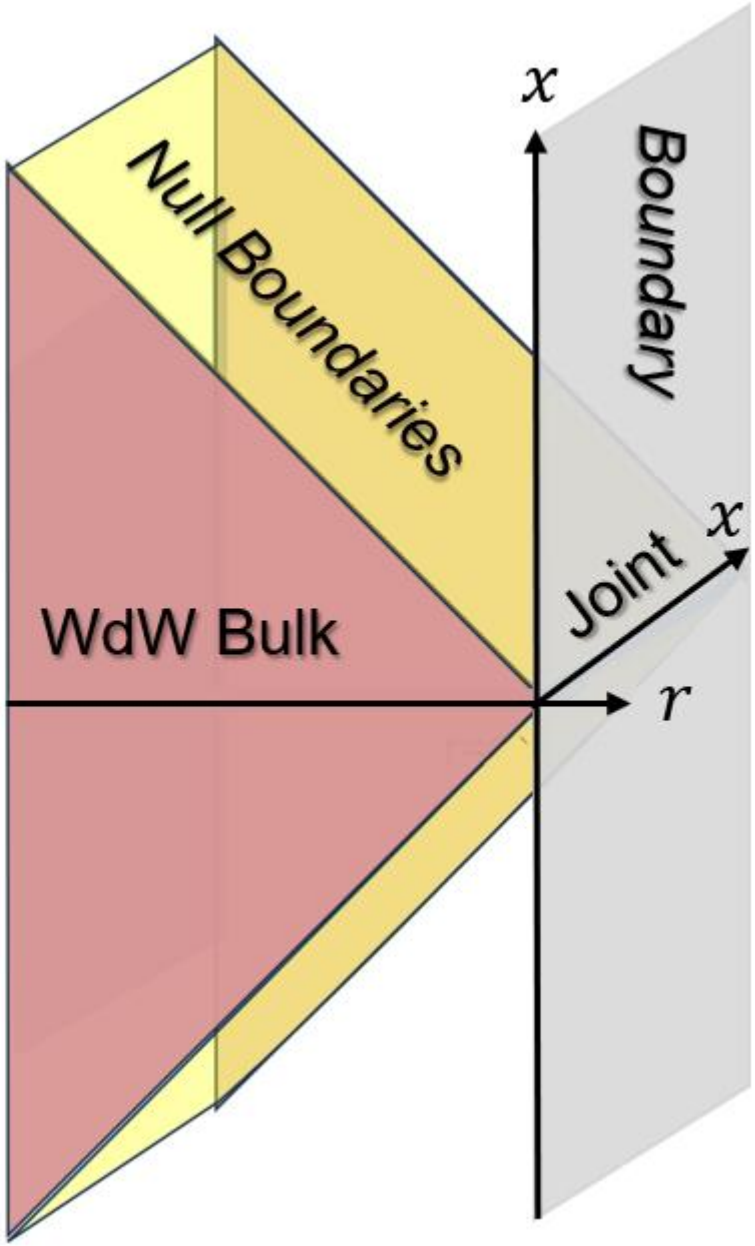
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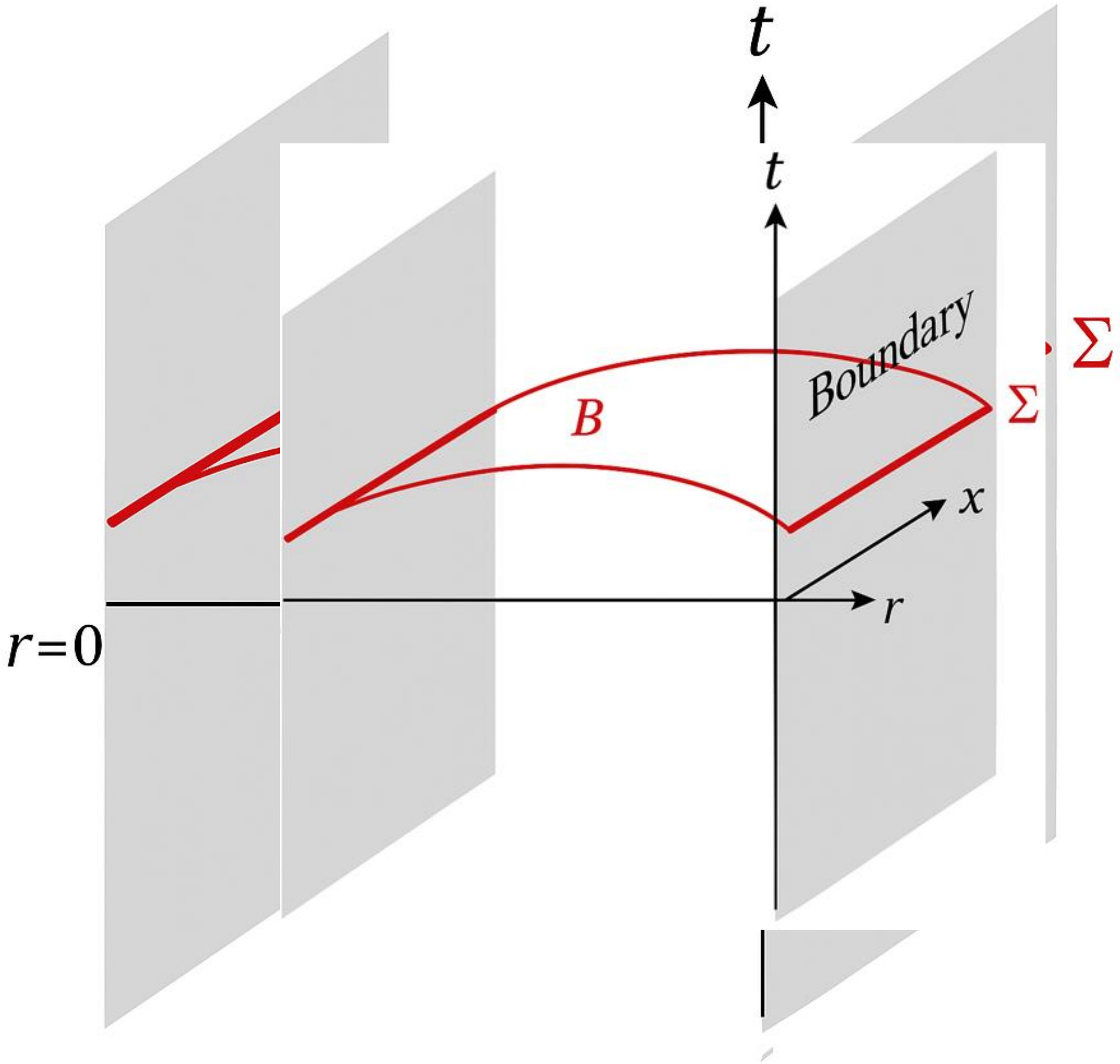
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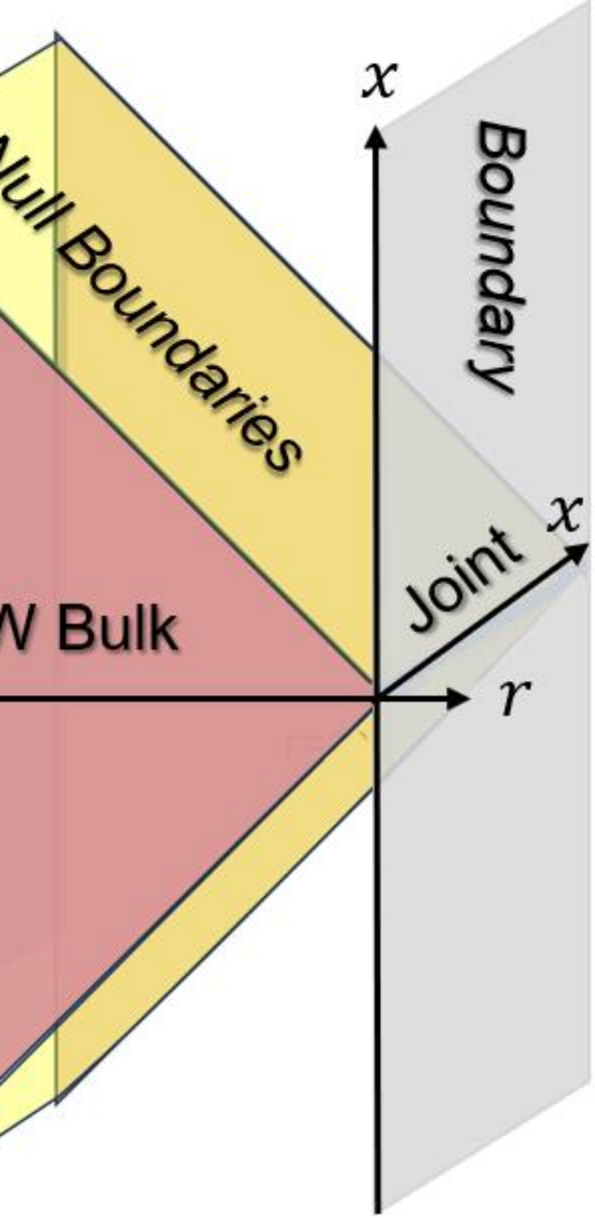
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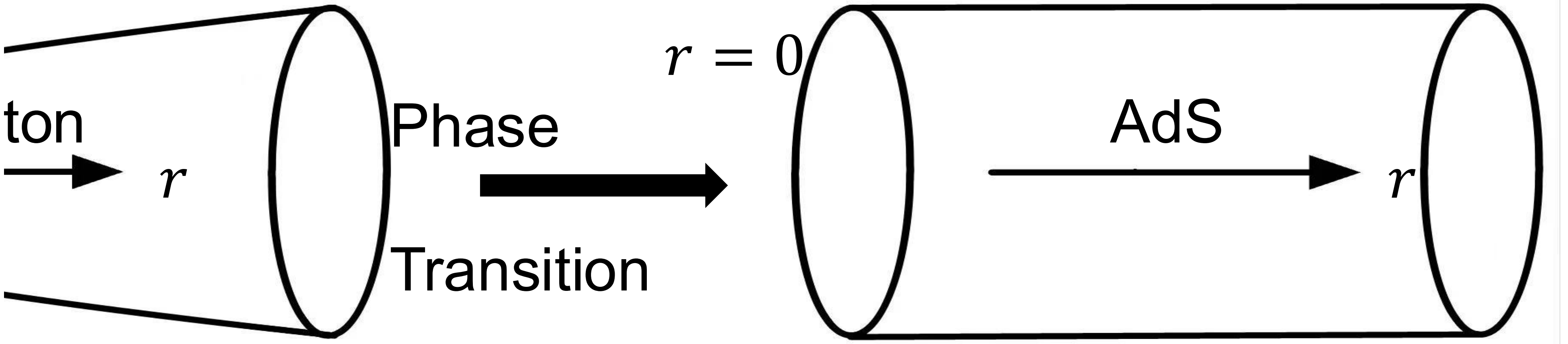
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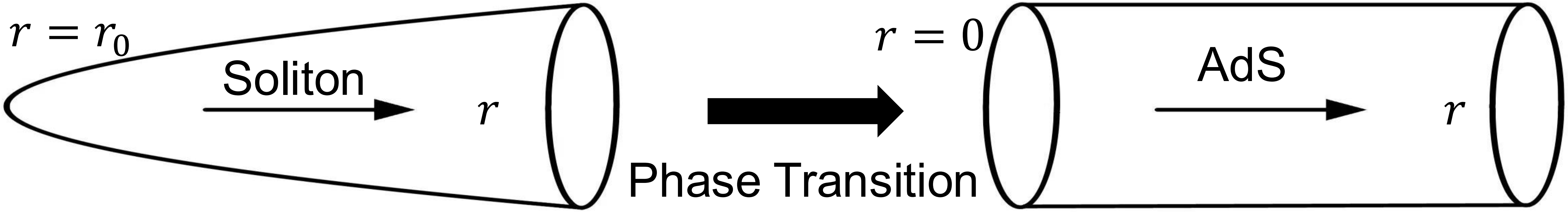

$r = r_0$
Soliton
r
Phase Transition
$r = 0$
AdS
r

Complexity=Action conjecture (CA)

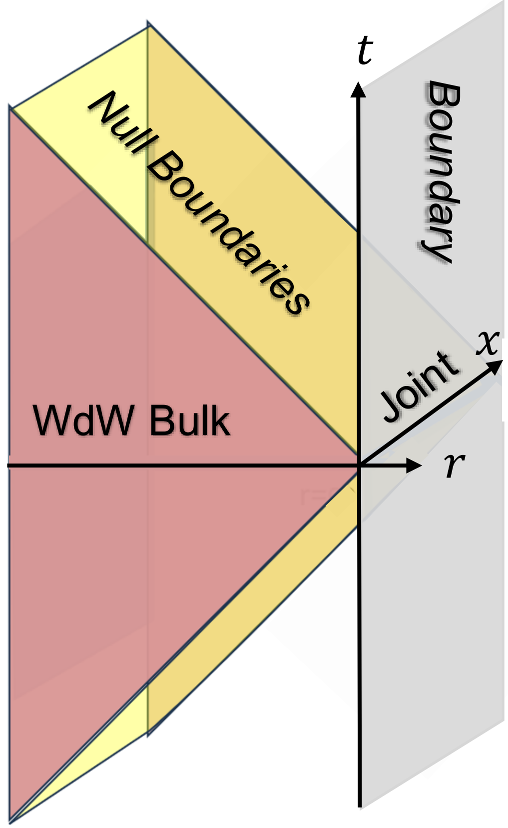
- CA Conjecture

The complexity of the boundary CFT state is proportional to the action of the Wheeler-de Witt patch

$$C_A = \frac{S}{\pi\hbar}$$

S is the action of the Wheeler-de Witt (WDW) patch, the union of all spacelike slices we considered before.

$$S = S_{bulk} + S_{bdy} + S_{joint}$$



Sketch of CA

arXiv:1509.07876 arXiv:1512.04993 (Brown, Roberts, Susskind, Swingle, Zhao 2015)

Complexity, scaling, and a phase transition

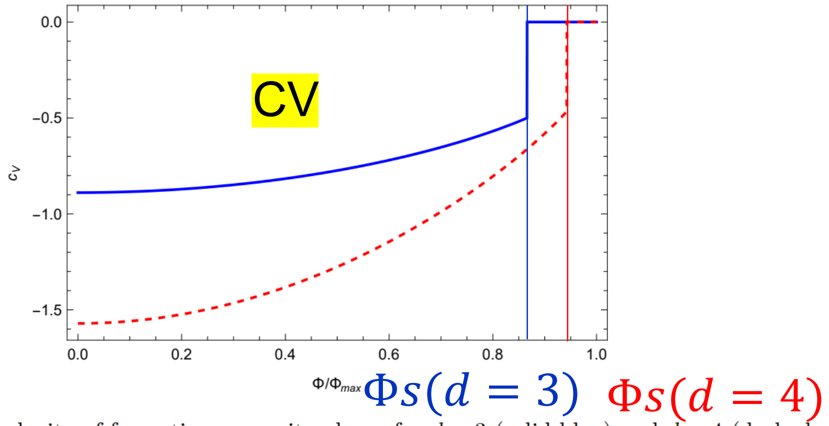


Figure 1. CV complexity of formation per unit volume for $d = 3$ (solid blue) and $d = 4$ (dashed red). Plotted curves are $c_V \equiv G\mathcal{C}_V(\Delta\phi/l)^{d-1}$, so complexity scales with an inverse power of $\Delta\phi$.

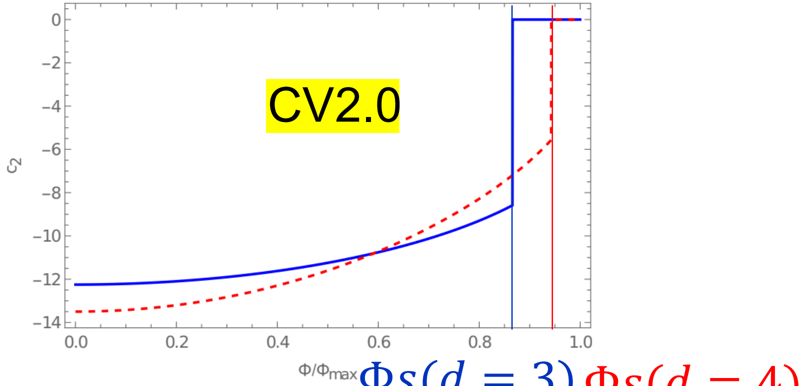
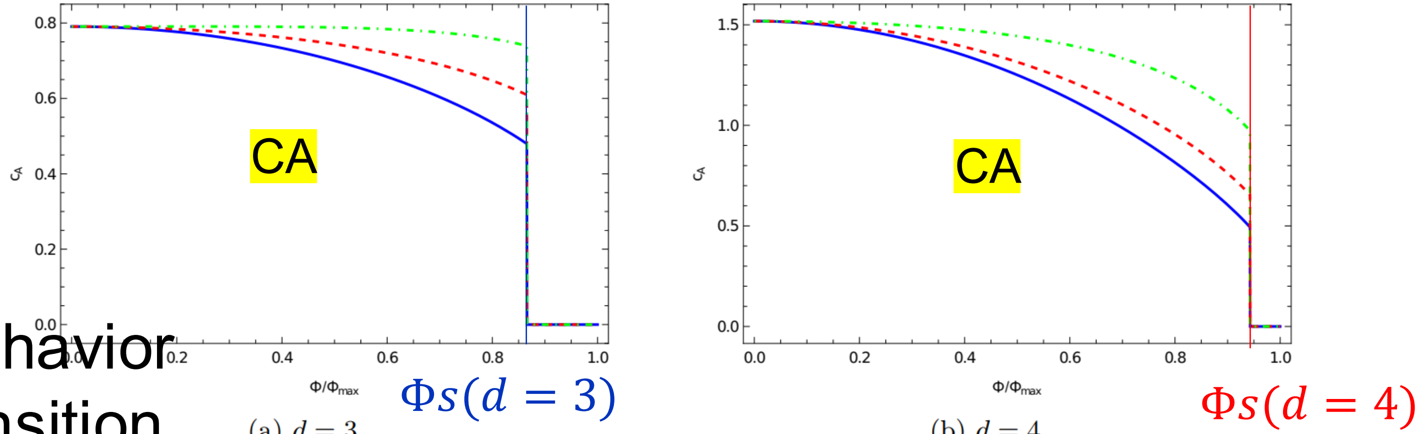


Figure 3. CV2.0 complexity of formation per boundary volume for $d = 3$ (solid blue) and $d = 4$ (dashed red). Plotted curves are $c_2 \equiv G\mathcal{C}_2(\Delta\phi/l)^{d-1}$, so complexity scales with an inverse power of $\Delta\phi$.



(a) $d = 3$. (b) $d = 4$.

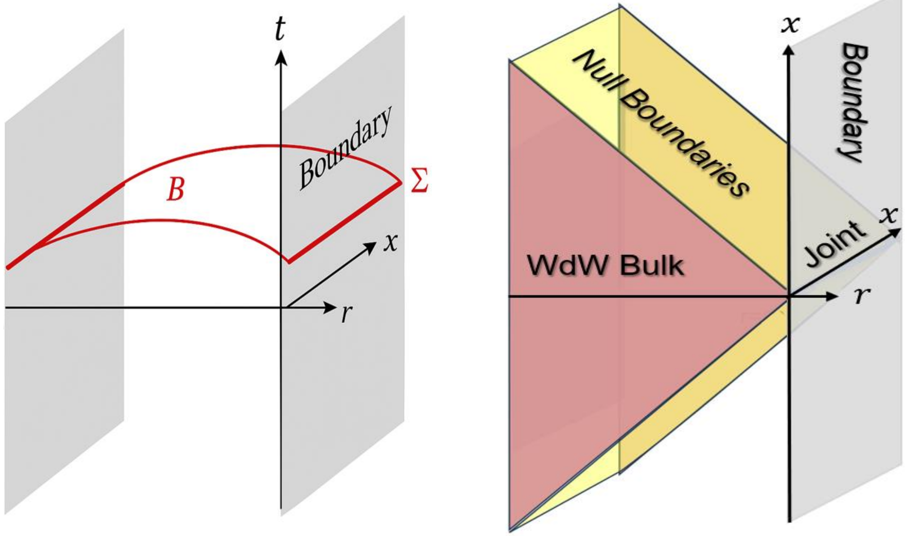
Figure 4. CA complexity of formation per unit boundary volume for $d = 3, 4$ as labeled with $\nu = 0$ (solid blue), $\nu = 1/(d-1)$ (dashed red), and $\nu = 1$ (dot-dashed green). Plotted curves are $c_A = G\mathcal{C}_A(\Delta\phi/l)^{d-1}$ as a function of the Wilson line Φ.

- Same scaling behavior
- Same phase transition

Complexity, scaling, and a phase transition

Jiayue Yang, and Andrew R. Frey (JHEP09(2023)029)

- Holographic complexity



- Same scaling behavior

$$\mathcal{C} \propto \Delta\phi^{-(d-1)}$$

- Same phase transition